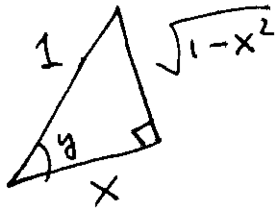
Math 1000 Review sheet Solutions

1) <u>Example</u>: To find $\frac{d}{dx} \arccos x$,

let $y(x) = \arccos x \iff x = \cos y$

$$\Rightarrow \quad 1 = -\frac{dy}{dx} \sin y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin y}$$



Now $x = \cos y$

$$\Rightarrow \quad \sin y = \sqrt{1-x^2}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

2) a) $$\frac{\sqrt{x+2}-3}{x-7} = \frac{\left(\sqrt{x+2}-3\right)\left(\sqrt{x+2}+3\right)}{(x-7)\left(\sqrt{x+2}+3\right)}$$

$$= \frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)} = \frac{1}{\sqrt{x+2}+3}$$

$$\longrightarrow \frac{1}{\sqrt{9}+3} = \frac{1}{6} \quad \text{as } x \to 7$$

b) For large x, $\dfrac{2x+1}{\sqrt{x^2+\sin(x)}} \sim \dfrac{2x}{\sqrt{x^2}}$

$\rightarrow 2$ as $x \to \infty$

c)
$$\lim_{x\to\infty} \left(\tan^{-1}(2x) - \frac{\pi}{2}\right) x$$

$$= \lim_{x\to\infty} \frac{\left(\tan^{-1}(2x) - \frac{\pi}{2}\right)}{\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+(2x)^2}\cdot 2}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-2x^2}{4x^2+1} = -\frac{1}{2}$$

d) $x \cot 2x = x\,\dfrac{\overbrace{\cos 2x}^{\sim 1}}{\underbrace{\sin 2x}_{\sim 2x}} \sim \dfrac{x\cdot 1}{2x}$ as $x \to 0$

$\rightarrow \boxed{\dfrac{1}{2}}$ as $x \to 0$.

e) 0

d) $-\infty$

3) $$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h}$$

$$= \frac{\left(\sqrt{1+2(x+h)} - \sqrt{1+2x}\right)\left(\sqrt{1+2(x+h)} + \sqrt{1+2x}\right)}{h\left(\sqrt{1+2(x+h)} + \sqrt{1+2x}\right)}$$

$$= \frac{1+2(x+h)-(1+2x)}{h\left(\sqrt{1+2(x+h)} + \sqrt{1+2x}\right)} =$$

$$= \frac{1}{\sqrt{1+2(x+h)} + \sqrt{1+2x}} \longrightarrow \frac{1}{\sqrt{1+2x}} \text{ as } h \to 0$$

$$\frac{1}{\sqrt{1+2x}} = f'(x).$$

4) If $f(x) = (\cos x)^{2x}$ then to find $f'(x)$,

write $f(x) = e^{\ln\left((\cos x)^{2x}\right)}$

$$= e^{2x \ln \cos x}$$

$$\Rightarrow f'(x) = e^{2x \ln \cos x} \cdot \frac{d}{dx}(2x \ln \cos x)$$

$$= e^{2x \ln \cos x}\, 2\left(\ln \cos x + x \frac{\sin x}{\cos x}\right)$$

5) $$y' \sec^2 y + \frac{1}{1+x^2} = 2y'yx + y^2$$

Plug in $x=0$, $y=\frac{\pi}{4}$:

$$y' \underbrace{\frac{1}{\left(\cos\frac{\pi}{4}\right)^2}}_{\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}=2} + 1 = \left(\frac{\pi}{4}\right)^2$$

$$\Rightarrow y' = \frac{\frac{\pi^2}{16} - 1}{2} = \frac{\pi^2}{32} - \frac{1}{2} .$$

6a) for Q5, when $x=0.1$, we use:

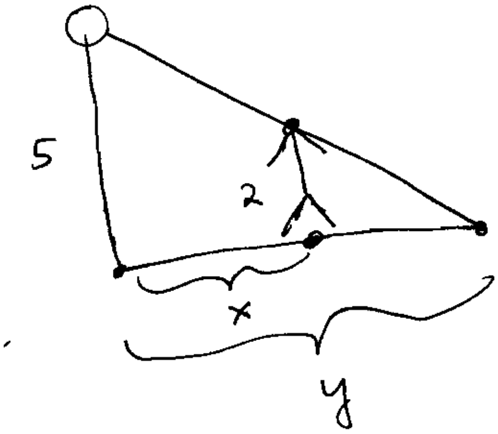
$$y(0.1) \approx y(0) + 0.1\, y'(0)$$
$$\approx \frac{\pi}{4} + 0.1\left(\frac{\pi^2}{32} - \frac{1}{2}\right) .$$

6b) To estimate $\ln 3$, note that $e = 2.7$ is close to 3, so we write:

$$\ln(3) = \underbrace{\ln e}_{1} + (3-e)\cdot\frac{1}{e}$$

$$= \frac{3}{e} .$$

7a)



Let x, y be as in diagram.

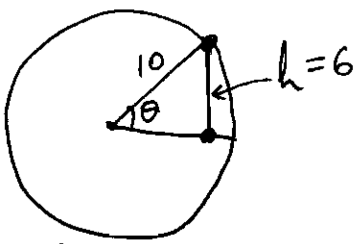
Then using similar triangles, we have:

$$\frac{y}{5} = \frac{y-x}{2} \Rightarrow \frac{y'}{5} = \frac{y'-x'}{2}$$

$$\Rightarrow y' = \frac{5}{3}x'. \quad \text{If } x' = \frac{1}{2} \text{ then}$$

$$\boxed{y' = \frac{5}{6}}.$$

7b)



$$\sin\theta = \frac{h}{10}$$

$$\Rightarrow \theta' \cos\theta = \frac{h'}{10}$$

$$\Rightarrow h' = 10\cos\theta \underbrace{\theta'}_{10}$$

$$= 100 \cdot \frac{4}{5} = 80 \ \frac{\text{meters}}{\text{min}}$$

$$\cos\theta = \frac{\sqrt{10^2 - h^2}}{10}$$

$$= \frac{\sqrt{64}}{10}$$

$$= \frac{4}{5}$$

7 c) $V = \pi r^2 l \quad ; \quad V' = 0$

$$V' = \pi \left(2 r r' l + r^2 l' \right) = 0$$

$$\Rightarrow \frac{r'}{r} = -\frac{1}{2} \frac{l'}{l}$$

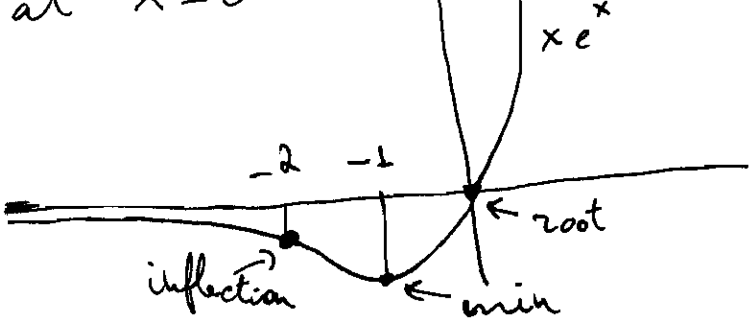
Now $\frac{l'}{l}$ is constant since $l' \propto l$

$\Rightarrow \frac{r'}{r}$ is constant.

8) a) $f(x) = x e^x \quad , \quad f' = (x+1) e^x ,$

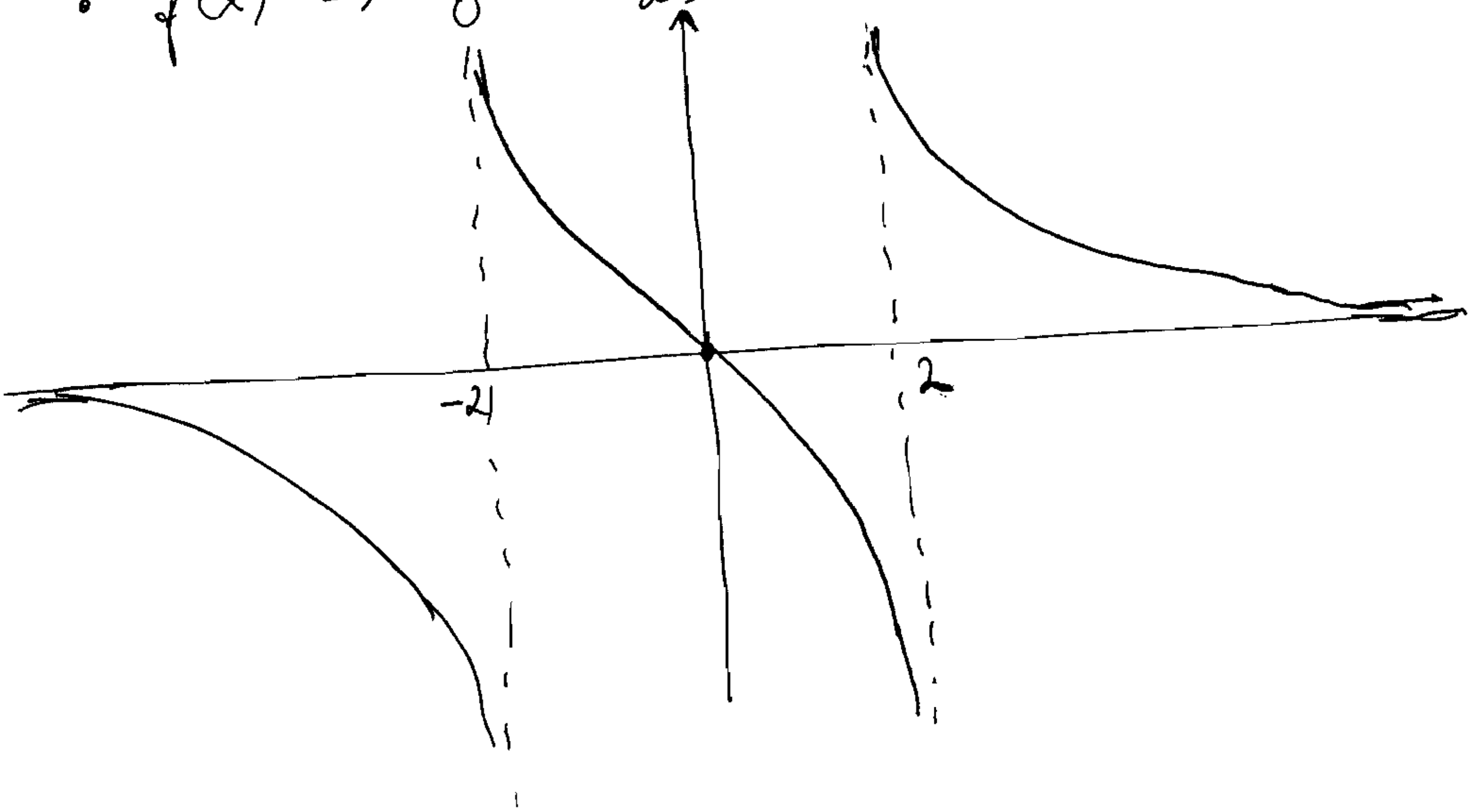
$$f'' = (x+2) e^x$$

Note:
- $f(x) \to \infty$ as $x \to \infty$
- $f(x) \to 0$ as $x \to -\infty$
- min at $x = -1$
- inflection at $x = -2 \Rightarrow \begin{cases} \text{convex if } x > -2 \\ \text{concave if } x < -2 \end{cases}$
- root at $x = 0$

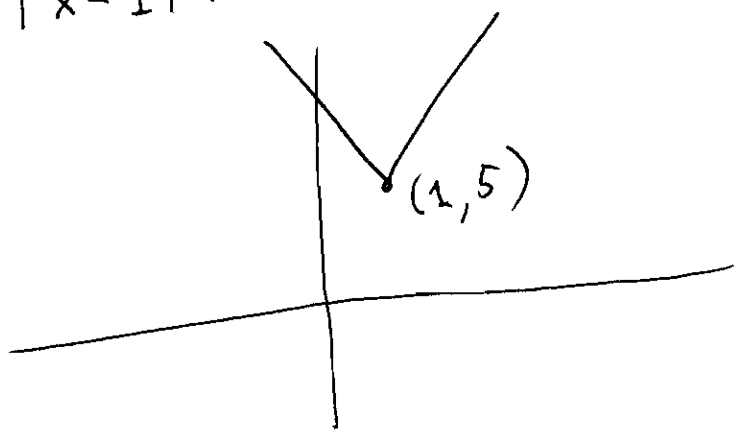


8b) $f(x) = \dfrac{x}{x^2-4} = \dfrac{x}{(x-2)(x+2)}$

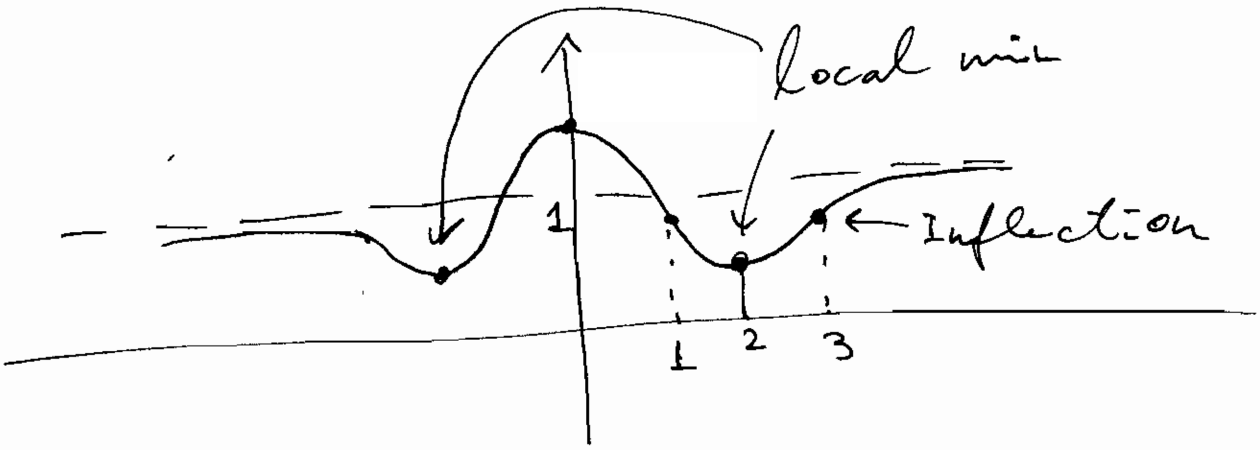
- Vertical asymptotes at $x = \pm 2$
- $f(x) \to 0^+$ as $x \to \infty$
- $f(x) \to 0^-$ as $x \to -\infty$



8c) $f(x) = 2|x-1| + 5$

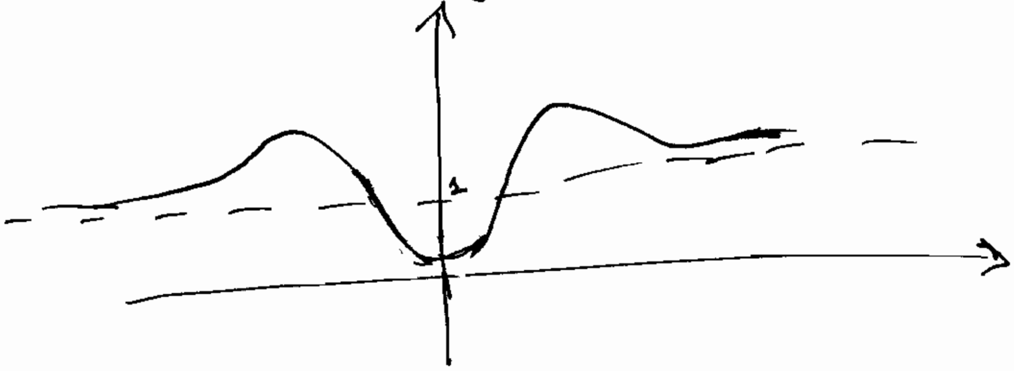


9)

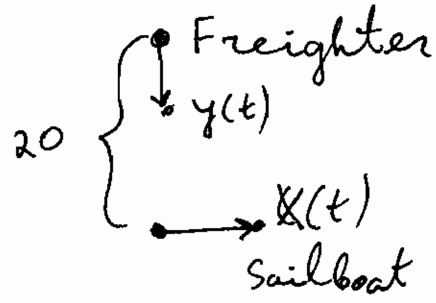


- local min at $x = \pm 2$, max at $x = 0$
- inflection at $x = \pm 1, \pm 3$

Another possibility:



10) a)



Let $x(t)$ be the position of sailboat at time t; let $y(t)$ be position of freighter.

Then $x(t) = 20t$; $y(t) = 20 - 40t$;

We want to find min $d(t)$

Where $d(t) = \sqrt{x^2 + y^2}$

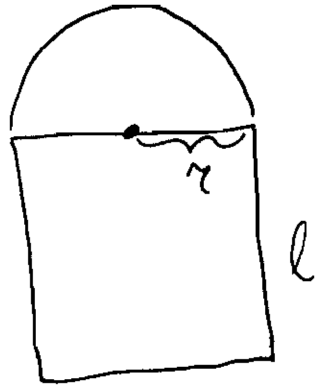
$$d^2 = (20t)^2 + (20 - 40t)^2$$
$$= 20^2 \left[t^2 + (1-2t)^2 \right]$$

$$\frac{d(d^2)}{dt} = 0 \Rightarrow \quad 2t + 2(1-2t)\cdot(-2) = 0$$
$$6t - 4 = 0$$
$$\boxed{t = \frac{2}{3}}$$

When $t = \frac{2}{3}$ hours [i.e. 12:40], the distance between the two ships is at its minimum, and is given by

$$d = 20\sqrt{\left(\frac{2}{3}\right)^2 + \left(1 - \frac{4}{3}\right)^2} = \frac{20}{\sqrt{3}}.$$

10b)



$$A = \frac{\pi r^2}{2} + 2rl$$

$$P = \pi r + 2l + 2r$$

$$= 2l + (2+\pi)r \Rightarrow l = \frac{P-(2+\pi)r}{2}$$

$$\Rightarrow A = \frac{\pi}{2}r^2 + \left(\frac{P-(2+\pi)r}{\cancel{2}}\right)\cancel{2}r$$

$$= \left(-\frac{\pi}{2} - 2\right)r^2 + Pr$$

$$A' = (-\pi - 4)r + P = 0$$

$$\Rightarrow \boxed{r = \frac{P}{4+\pi}}$$

$$l = \frac{P}{2}\left(1 - \frac{2+\pi}{4+\pi}\right) = \frac{P}{4+\pi}$$

Note that A is a (concave) parabola

so $\boxed{l = r = \frac{P}{4+\pi}}$ is the

<u>absolute max</u>.

11) $V(t) = V(0) + \int_0^t a(s)\,ds = 3 + \int_0^t \sqrt{s}\,ds$

$= 3 + \frac{2}{3} t^{3/2}$

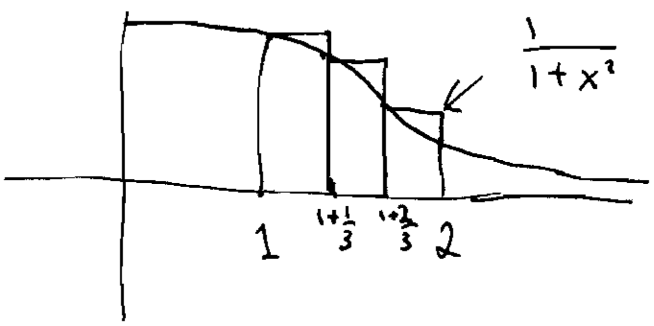
$p(t) = p(0) + \int_0^t V(s)\,ds$

$= 2 + \int_0^t \left(3 + \frac{2}{3} s^{3/2}\right) ds$

$= 2 + 3t + \frac{2}{3} \cdot \frac{2}{5} t^{5/2}$

$\Rightarrow p(1) = 2 + 3 + \frac{4}{15} = 5 + \frac{4}{15}$

12)



$\int_1^2 \frac{1}{1+x^2}\,dx$

$X_0 = 1$

$X_1 = 1 + \frac{1}{3} = \frac{4}{3}$

$X_2 = 1 + \frac{2}{3} = \frac{5}{3}$

$X_3 = 2$

$\Delta x = \frac{1}{3}$

$\approx \frac{1}{3}\left(\frac{1}{1+{x_1^*}^2} + \frac{1}{1+{x_2^*}^2} + \frac{1}{1+{x_3^*}^3}\right)$ $\quad x_1^* = 1,\ x_2^* = \frac{4}{3}\ \ x_3^* = \frac{5}{3}$

$\approx \frac{1}{3}\left(\frac{1}{1+1} + \frac{1}{1+\frac{16}{9}} + \frac{1}{1+\frac{25}{9}}\right) = 0.375.$

13) Note that for $x \geq 0$,

$$x^3 \leq x^3 + 1$$

$$\Rightarrow \quad \frac{1}{x^3+1} \leq \frac{1}{x^3}$$

$$\Rightarrow \quad \int_1^2 \frac{1}{x^3+1}\,dx \leq \int_1^2 x^{-3}\,dx = -\frac{1}{2}x^{-2}\Big|_1^2$$

$$= \frac{1}{2}\left(1-\frac{1}{4}\right)$$

$$= \frac{3}{8}.$$

14)

a) $$\int \frac{x^2+\sqrt{x}}{x}\,dx = \int\left(x + x^{-\frac{1}{2}}\right)dx$$

$$= \frac{x^2}{2} + 2x^{\frac{1}{2}}.$$

b) $$\int_{x=0}^{x=\pi/2} \frac{\cos x}{1+\sin x}\,dx$$

$u = 1+\sin x$

$du = \cos x \, dx$

$x = 0 \Rightarrow u = 1$

$x = \frac{\pi}{2} \Rightarrow u = 1+1 = 2$

$$= \int_{u=1}^{u=2} \frac{du}{u} = \ln u\Big|_1^2$$

$$= \ln 2.$$

c) $\frac{1}{2}e^{2x} + C$

d) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^1$

$= \underbrace{\arctan 1}_{\frac{\pi}{4}} - \underbrace{\arctan 0}_{0}$

$= \frac{\pi}{4}$

e) $\displaystyle\int_{x=0}^{x=4} \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx \Rightarrow \frac{dx}{\sqrt{x}} = 2\,du$

$x = 0 \Rightarrow u = 1$

$x = 4 \Rightarrow u = 3$

$= \displaystyle\int_{u=1}^{u=3} u^3\, 2\, du = 2\,\frac{u^4}{4}\Big|_1^3 = \frac{1}{2}\left(3^4 - 1\right)$

$= 40$

f) $f(x) = \sin x^3$ is an <u>odd function</u> i.e. $f(x) = -f(-x)$

$\Rightarrow \displaystyle\int_{-2}^{2} f(x) = 0$

15) $g'(x) = 2x\left(\frac{1}{\sqrt{x^2} + (x^2)^2}\right) = \frac{2x}{x + x^4}$

$= \frac{2}{1 + x^3}$

16) a) Let $f(x) = x^3 - 1 - x$.

Note that $f(0) = -1 < 0$ and

$f(10) \approx 1000 > 0$

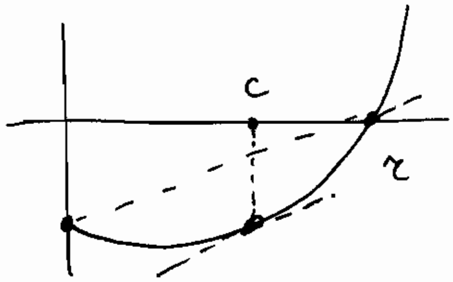
so f has at least one root $x > 0$.

Also note that $f'' = 6x > 0$ for all $x > 0$

So $f(x)$ is convex for $x > 0$.

Claim: Any convex function with $f(0) < 0$ can have at most one root $x > 0$.

Proof of claim: If $f(x)$ has no roots then we are done. So assume $f(x)$ has at least one root and let r denote the <u>smallest</u> such root.



Then by the Mean Value Theorem, there exists a point $c \in [0, r]$ with

$$f'(c) = \frac{f(r) - f(0)}{r - 0} = \frac{-f(0)}{r} > 0$$

$\Rightarrow f'(c) > 0.$

But $f'' \geqslant 0 \Rightarrow f'$ is increasing

$\Rightarrow f'(x) \geqslant f'(c)$ whenever $x > c$

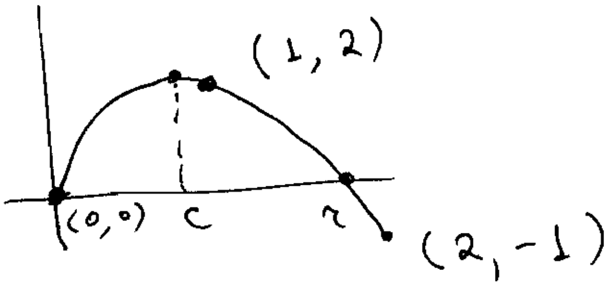
$\Rightarrow f'(x) > 0$ for all $x > c$

$\Rightarrow f$ is increasing for all $x > c$

$\Rightarrow f$ has at most one root on $[c, \infty]$

But we assumed that r is the smallest root of $f(x)$ and since $c < r$, it is the only one. ∎

16 b)



By IVT, there exists $r \in [1, 2]$ with $f(r) = 0 = f(0)$

Then by Rolle's theorem, there exists $c \in (0, r)$ with $f'(c) = 0$ $\blacksquare$

16 c) Let $f(x) = \sin x - x$.

We need to show that $f(x) \leq 0$ for $x \geq 0$.

Note that $f(0) = 0$ and

$$f'(x) = \cos x - 1 \leq 0$$

[since $-1 \leq \cos x \leq 1$]

$\Rightarrow$ f is <u>decreasing</u>

$\Rightarrow$ $f(x) \leq f(0)$ for $x > 0$

$\Rightarrow$ $f(x) \leq 0$ $\blacksquare$

Math 1000 Review questions

Final exam: 11 December, 8:30AM, in Dalplex. NO NOTES, NO CALCULATORS!!!

1. You need to know basic facts about these functions:

$$f(x) = \frac{1}{x},\ x^p,\ e^x,\ \sin x,\ \cos x,\ \tan x$$

and their inverses: $f(x) = \frac{1}{x},\ x^{1/p},\ \ln(x),\ \sin^{-1} x,$ etc. The basic facts include:

- Domain and range; various asymptotes and graphs (example: what happens to $\ln x$ as $x \to 0$, or as $x \to \infty$?),
- Basic values (e.g. what is $\ln(1), \ln(e), \arctan(1)$)
- Their derivatives. For inverse trigs, you need to know how to find these.

2. Limits: compute

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}; \quad \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+\sin(x)}}; \quad \lim_{x\to\infty} \left(\tan^{-1}(2x) - \frac{\pi}{2}\right) x;$$
$$\lim_{x\to 0} (x \cot(2x)); \quad \lim_{x\to\infty} e^{-x} x^{2006}; \quad \lim_{x\to 1^+} \ln(x-1).$$

3. Differentiation from first principles: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Example: Find the derivative of $f(x) = \sqrt{1+2x}$ using the definition of derivative. No credit will be given if you use differentiation rules but you can use them to check your answer.

4. Basic differentiation for example $\frac{d}{dx}\tan(x\cos 2x)$, inverse trigs (example: $\frac{d}{dx}\tan^{-1}(x)$), log differentiation (example: $\frac{d}{dx}\left((\cos x)^{2x}\right)$).

5. Implicit functions: Suppose that $\tan y + \arctan x = y^2 x + 1$. Find $\frac{dy}{dx}$ at the point $x = 0,\ y = \frac{\pi}{4}$.

6. Linear approximation:

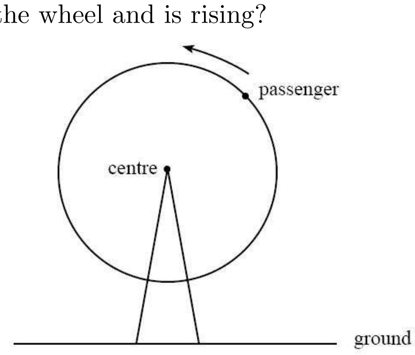
   (a) For question 5, estimate y when $x = 0.1$.

   (b) Estimate $\ln(3)$.

7. Related rates:

   (a) A man 2 m tall walks toward a lamppost on level ground at a rate of 0.5 m/s. If the lamp is 5 m high on the post, how fast is the length of the man's shadow decreasing when he is 3 m from the post? How fast is the shadow of his head moving at that time?

   (b) A circular ferris wheel with radius 10 metres is revolving at the rate of 10 radians per minute. How fast is a passenger on the wheel rising when the passenger is 6 metres higher than the centre of the wheel and is rising?



   (c) A lump of clay is being rolled out so that it maintains the shape of a circular cylinder (and its volume remains constant). If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself. Note: $V = \pi r^2 l$.

8. Graphing: Sketch the graph of a given function $f(x)$. Indicate behaviour as $x \to \pm\infty$, any vertical asymptotes, roots, critical points, max/min, concavity. For example, $f(x) = xe^x$; $f(x) = \frac{x}{x^2-4}$; $f(x) = 2|x-1| + 5$.

9. More graphing: Suppose $f(x)$ is an even function ($f(x) = f(-x)$), continuous and differentiable everywhere, and has the following properties:

   - It has two inflection points for $x > 0$ at $x = 1$ and $x = 3$ and $f'(2) = 0$;
   - $f(x) > 0$ for all x and $f(x) \to 1$ as $x \to \infty$.

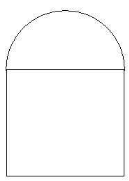
   Sketch the graph of f. Indicate any critical points, max/min, concavity. Does $f(x)$ have any roots?

10. Max/min:

    (a) Find the absolute minimum and the absolute maximum value of $f(x) = \frac{x}{x^2+1}$ on the interval $[-\frac{1}{2}, 3]$.

    (b) At noon, a sailboat is 20 km due south of a freighter. The sailboat is travelling due east at 20 km/hr, and the freighter is travelling due south at 40 km/hr. When are the two ships closest to each other, and what is the distance between them at that point?

    (c) A window has the shape of a rectangle surmounted by a semicircle (see diagram). For a fixed perimeter P, what should the dimensions of the rectangle and semicircle be, in order to maximize the area of such a window?



11. A particle is moving on a straight line and its acceleration at time t is given by $a(t) = \sqrt{t}$. At time $t = 0$ its position is given by $p(0) = 2$ and its velocity is $v(0) = 3$. Find the the position of the particle at time $t = 1$.

12. Set up the Riemann sum to approximate the area under the curve $y = \frac{1}{x^3+1}$ between $x = 1$ and $x = 2$ with $n = 3$ subintervals and with the sample points taken at the left endpoints. Sketch a diagram to illustrate.

13. Show that $\int_1^2 \frac{1}{x^3+1} dx \leq 3/8$. (hint: first show that $\frac{1}{x^3+1} \leq \frac{1}{x^3}$)

14. Evaluate the following:

$$\int \frac{x^2+\sqrt{x}}{x} dx; \quad \int_0^{\pi/2} \frac{\cos x}{1+\sin(x)} dx; \quad \int e^{2x} dx; \quad \int_0^1 \frac{1}{x^2+1} dx; \quad \int_0^4 \frac{(\sqrt{x}+1)^3}{\sqrt{x}} dx;$$

$$\int_{-2}^{2} \sin\left(x^3\right) dx.$$

15. Suppose that $g(x) = \int_1^{x^2} \frac{1}{\sqrt{t}+t^2} dt$. Find $g(1)$ and $g'(1)$.

16. IVT, MVT etc:

    (a) Using the Intermediate Value Theorem, show that there is at least one positive solution to $x^3 - 1 = x$. Then show that that there is exactly one such solution with $x > 0$ using either Rolle's or Mean Value Theorem.

    (b) A function $f(x)$ satisfies $f(0) = 0$, $f(1) = 2$ and $f(2) = -1$. It is known that f is differentiable everywhere. Show that $f'(c) = 0$ for some number c. Give complete justification, specifying any relevant theorems.

    (c) Show that $\sin(x) < x$ for all $x > 0$.