Math 1010 Sample final exam questions

1. Evaluate the following integrals:

$$\text{(a)} \int \left(\frac{1}{x^{3/2}} + \frac{1}{1+x^2} + \sin(2x) \right) dx \qquad \text{(b)} \int \frac{\ln x}{\sqrt{x}}\, dx$$

$$\text{(c)} \int_0^1 \arctan x \, dx \qquad \text{(d)} \int_0^{\pi/4} \sin^3(2x)\, dx \qquad \text{(e)} \int x e^{2x}\, dx$$

$$\text{(f)} \int \frac{1}{(4+x^2)^{3/2}}\, dx \qquad \text{(g)} \int \frac{x}{(x+1)^2 (x-1)} \qquad \text{(h)} \int \frac{x^3}{\sqrt{x^2+4}}\, dx$$

2. Consider the region R bounded by the curves $y = \sin x$, $y = \cos x$, $y = 0$, with $\frac{\pi}{4} \leq x \leq \pi$.

   (a) Sketch the region R.

   (b) Find its area.

   (c) Set up the integral for the volume of the solid that is obtained by rotating R around the y axis. DO NOT evaluate the resulting integral.

   (d) Set up the integral for the volume of the solid that is obtained by rotating R around the x axis. DO NOT evaluate the resulting integral.

3. (a) A paper mill would like to know how much wood is inside a tree. To find out, a worker measures the diameter d of the tree at certain heights x and records it in a table as follows:

x	0	5	10	15	20
d	0.6	0.6	0.5	0.4	0.4

   Use the Simpson's rule to estimate the volume of this tree. Hint: the area of a circle of diameter d is given by $\frac{\pi}{4} d^2 = 0.8 d^2$.

   (b) Suppose that $f(x)$ is defined as $f(x) = \int_1^x \frac{e^t}{t}\, dt$. If asked to find $\int_1^2 f(x)\, dx$ using the Trapezoid rule, how large should you choose n to be sure that the error is at most 0.1? Recall that the error for the Trapezoid rule is bounded by $\frac{(b-a)^3 \max |f''(x)|}{12n^2}$.

4. Determine whether the following integrals are convergent or divergent:

$$(a) \int_2^{\infty} \frac{\ln x}{\sqrt{x} - 1}\, dx \qquad (b) \int_1^7 \frac{e^{-x}}{\sqrt{x-1}}\, dx$$

5. (a) Set up an integral for the length of the curve $y = \sin x$, $0 \leq x \leq 2\pi$. DO NOT evaluate the resulting integral

   (b) Set up an integral for the length of the ellipse given in parametric coordinates by $x = \cos t$, $y = \sqrt{2} \sin t$. DO NOT evaluate the resulting integral.

   (c) Show that the lengths you get in parts (a) and (b) are equal (integration is not required).

6. (a) Find the solution to the differential equation $x^3 \frac{dy}{dx} = y^2$ with initial condition $y(1) = 1$.

   (b) The population of a small country was 7 million in 1990 and 8 million in 2000. Assuming the exponential growth of the form $y = Ae^{kt}$, in what year will the population reach 9 million?

7. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{5(-3)^n}{4^{n+1}}.$$

8. Determine whether the following series converge or diverge, and give a reason why.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{2n+1}} \qquad (b) \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$$

$$(c) \sum \frac{n+2}{3n} \qquad (d) \sum_{n=1}^{\infty} \frac{4^n}{n!}.$$

9. (a) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n}$.
   (b) Find the interval of convergence of the series in part (a).

10. (a) Write down the Taylor series for $y = \sin x$ centered at $x = 0$.
    (b) Find the Taylor series for $y = \frac{1}{x^2} \sin\left(x^2\right)$.
    (c) Using results of part (b), find an expression for $\int_0^1 \frac{1}{x^2} \sin\left(x^2\right) dx$.

11. Find the Taylor series of $f(x) = \frac{1}{2x+1}$ centered at $x = 0$.

12. (a) Sketch the graph of $y = 2\cos(2x)$ in cartesian coordinates.
    (b) Sketch the graph of $r = 2\cos(2\theta)$ in polar coordinates.
    (c) Sketch the the region bounded by the curves $r = 2\cos(2\theta)$, $r = 1$, with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and with $r \leq 1$.
    (d) Find the area of the region in part (c).