Homework 1

1. Evaluate the following limits or explain why it doesnt exsit.

$$\text{(a)}\quad \lim_{x\to 2}\frac{x-2}{x^2-4}$$

$$\text{(b)}\quad \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$

$$\text{(c)}\quad \lim_{x\to 1}\frac{x^3-1}{x-1}$$

$$\text{(d)}\quad \lim_{t\to -1}\frac{t^2}{t+1}$$

2. Find the following limits (they may be infinite).

$$\text{(a)}\quad \lim_{x\to\infty}\frac{x^3+x-5}{3+3x^3}$$

$$\text{(b)}\quad \lim_{x\to-\infty}\frac{x^3+x-5\sin x}{3+3x^3}$$

$$\text{(c)}\quad \lim_{x\to\infty}\frac{x^2}{\sqrt{3+3x^3}}$$

$$\text{(d)}\quad \lim_{x\to\infty}\frac{x^{3/2}}{\sqrt{3+3x^3}}$$

$$\text{(e)}\quad \lim_{x\to\infty}\frac{x}{\sqrt{3+3x^3}}$$

$$\text{(f)}\quad \lim_{x\to\infty}\frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{4x+2}}$$

$$\text{(g)}\quad \lim_{x\to\infty}\sqrt{x+1}-\sqrt{x}$$

3. (a) Consider

$$\lim_{x\to 0^+}\frac{\left(x^{1/2}+2x^2\right)^{1/2}}{x^p}.$$

Find p such that this limit exists and is non-zero. What is its value?

(b) Repeat part (a) but with "$x\to 0^+$" replaced with "$x\to+\infty$".

4. (a) Consider the function

$$f(x)=\frac{1}{1-x^2}.$$

Find the limits

$$\lim_{x\to 1^+}f(x),\quad \lim_{x\to 1^-}f(x),\quad \lim_{x\to\infty}f(x)$$

Sketch the graph of the function $f(x)$ (note the following useful property: $f(x)=f(-x)$). Indicate vertical and horizontal asymptotes.

(b) Consider the function

$$f(x)=\frac{1}{(1-x)^2}.$$

Find the limits

$$\lim_{x\to 1^+}f(x),\quad \lim_{x\to 1^-}f(x),\quad \lim_{x\to\infty}f(x)$$

Sketch the graph of the function $f(x)$. Indicate vertical and horizontal asymptotes.

5. Suppose that $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $x \in (-1,1)$. Use the Squeeze theorem to prove that $\lim_{x\to 0} f(x) = \sqrt{5}$.

6. Use the Squeeze theorem to evaluate $\lim_{x\to 0} x\cos(1/x)$.

7. Here are two methods to compute $\sqrt{2} \approx 1.414213562$ numerically. The first is called "Newton iteration". Start with $x_0 = 2$ and define, iteratively,

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}. \tag{1}$$

The second method is to start with $x_0 = 2$ and define, iteratively,

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}. \tag{2}$$

   (a) Starting with $x_0 = 2$, apply the iteration (1) several times (using calculator/computer). Record your answers to at least 8 decimal digits. How many iterations did you need to get 8 decimal places correctly?

   (b) Starting with $x_0 = 2$, apply the iteration (2) 10 times. How many correct decimal places did you get?

   (c) Based on your observations in parts (a) and (b), estimate how many iterations you will need to get 1000 decimal places of $\sqrt{2}$ using either (1) or (2).

   (d) Solve the equation $x = \frac{x+2}{x+1}$. What does this tell you about (2)?

8. [BONUS] The set of all algebraic numbers $\mathcal{A}$ is defined to be a set of all roots of all polynomials with integer coefficients. That is,

$$\mathcal{A} = \{x : a_n x^n + a_{n-1}x^{n-1} + ... + a_0 = 0, \quad n \in \mathbb{N},\ a_0, a_1, ...a_n \in \mathbb{Z}.\}$$

   (a) Show that $\sqrt{2}, \sqrt{3}+\sqrt{2}, \left(\sqrt{3}+\sqrt{2}+1\right)^{1/3} \in \mathcal{A}$.

   (b) Show that $\mathcal{A}$ is a countable set.

   Note: a real number that is not algebraic is called transcendental. Since $\mathcal{A}$ is countable but $\mathbb{R}$ is not, there are uncountably many transcendental numbers. Despite their abundance, it is difficult to come up with even one simple example of a transcendental number.

9. [BONUS] Show that there uncountably many letters "L", all of the same size, that can be fit into a plane without intersecting each-other. Show that there is only countably many letters "O" for which this can be done. What about the letter "X"? What if you are allowed to resize it?