Homework 2

Recall the rigorous definition of what it means that a sequence $a_n \to l$ (i.e. $\lim_{n\to\infty} a_n = l$):

- "For every given $\varepsilon > 0$, there exists N (which depends on the choice of ε) such that $|a_n - l| < \varepsilon$ for all $n \geq N$."

1. Consider the constant sequence $1, 1, 1, \ldots$ (i.e. $a_n = 1$ for all n).

   (a) Show, using the rigorous definition, that $\lim_{n\to\infty} a_n = 1$.

   (b) Show, using the rigorous definition, that $\lim_{n\to\infty} a_n \neq 0$.

2. (a) Let $x_n = 2^{-n}$. Show that $x_n \to 0$ as $n \to \infty$.

   (b) Let $x_0 = 0.1$ and and define iteratively, $x_{n+1} = (0.4 + x_n)\, x_n$ for $n \geq 0$. Compute $x_1, x_2, .$ Show rigorously that $x_n \to 0$.

3. Show that if $a_n \to l$ with $l > 0$ then $1/a_n \to 1/l$.

4. [BONUS] The *logistic map* is defined iteratively by $x_{n+1} = r x_n (1 - x_n)$ where r is a parameter. Take $x_0 = 0.5$ for simplicity.

   (b) Fix $r = 1.5$, and compute the first 10 iterates of x_n. You can use a computer for this. What is the limit $\lim_{n\to\infty} x_n$ for this value of r?

   (c) Repeat part (b) for $r = 2,\ 2.5,\ 3,\ 3.2,\ 3.5,\ 3.52$. Comment on anything interesting that you observe.

   (d) You should see that something weird happens at $r = 3$. For $r \in (1, 3)$, based on your observations from part (c), can you conjecture what is the limit $\lim_{n\to\infty} x_n$ as a function of r?

   (e) Conjecture what happens when $r \in (3, 3.5)$.

5. [BONUS]: Consider a sequence $x_{n+1} = x_n^{x_n}$, with $x_0 = 0.5$. Using a computer, convince yourself that $x_n \to 1$ *very slowly.* Using a computer, determine how big should n be, so that $|x_n - 1| < 0.01$? so that $|x_n - 1| < 0.001$? Based on your observations, guess how big should n be so that $|x_n - 1| < \varepsilon$ for any given (small) ε?

6. Let $f(x) = \frac{1}{2-x}$.

   (a) Find a number $\delta > 0$ such that $|f(x) - f(1)| \leq 0.1$ whenever $|x - 1| < \delta$.

   (b) Given an $\varepsilon > 0$, find a number $\delta > 0$ such that $|f(x) - f(1)| \leq \varepsilon$ whenever $|x - 1| < \delta$.

   (c) Conclude that $f(x)$ is continuous at $x = 1$.

7. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions on the interval $[-4, 4]$.

   Solution. Let $f(x) = x^3 - 15x + 1$. Then $f(-4) < 0,\ f(0) > 1,\ f(1) < 0$ and $f(4) > 0$. Hence my intermediate value theorem, there is a root in $(-4, 0)$, another root in $(0, 1)$ and a third root in $(1, 4)$.

8. Show that the function $f(x) = \sin(x - a)\sin(x - b) + x$ has the value $(a + b)/2$ at some point x.

9. Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Show that there is a number $c \in [0, 1]$ such that $f(c) = c$.

10. Prove that at any instant in time, there exist two points on the equator that have the same temperature and that are antipodal to each other. (hint: you may assume that the temperature is a continuous function of space).