Homework 2

Recall the rigorous definition of what it means that a sequence $a_n \to l$ (i.e. $\lim_{n\to\infty} a_n = l$):

- "For every given $\varepsilon > 0$, there exists N (which depends on the choice of ε) such that $|a_n - l| < \varepsilon$ for all $n \geq N$."

1. Consider the constant sequence $1, 1, 1, \ldots$ (i.e. $a_n = 1$ for all n).

   (a) Show, using the rigorous definition, that $\lim_{n\to\infty} a_n = 1$.

   (b) Show, using the rigorous definition, that $\lim_{n\to\infty} a_n \neq 0$.

   Solution. (a) Let $a_n = 1$. Then $|a_n - 1| = 0$ for all n. So given any $\varepsilon > 0$, let $N = 1$. Then $|a_n - 1| < \varepsilon$ for all $n \geq 1$.

   (b) Let $\varepsilon = 0.5$. Note that $|a_n - 0| = 1 > 0.5$ *for all* n. So then there exists no N such that $|a_n - 0| < 0.5$ for all $n \geq N$.

2. (a) Let $x_n = 2^{-n}$. Show that $x_n \to 0$ as $n \to \infty$.

   (b) Let $x_0 = 0.1$ and and define iteratively, $x_{n+1} = (0.4 + x_n)\, x_n$ for $n \geq 0$. Compute $x_1, x_2, .$ Show rigorously that $x_n \to 0$.

   Solution. (a) Given $\varepsilon > 0$, let N be such that $2^{-N} < \varepsilon$. Then $2^{-n} < \varepsilon$ for all $n \geq N$, since $2^{-n} \leq 2^{-N}$ whenever $n \geq N$.

   (b) To show $x_n \to 0$, note that $x_1 = (0.4 + 0.1)x_0 = 0.05 \leq 0.1$, then $x_2 = (0.4 + x_1)x_1 \leq (0.4+0.1)x_1 \leq 0.5x_1$ and similarly, $x_{n+1} \leq 0.5x_n$ for all $n \geq 1$. This means that $0 \leq x_n \leq 0.1 \times 2^{-n}$. Since $0.1 \times 2^{-n} \to 0$ and x_n is "squeezed" between 0 and 0.1×2^{-n}, x_n also $\to 0$.

3. Show that if $a_n \to l$ with $l > 0$ then $1/a_n \to 1/l$.

   Solution. $\left|\frac{1}{a_n} - \frac{1}{l}\right| = \frac{|a_n - l|}{|a_n l|}$. Since $a_n \to l > 0$, there exists N_1 such that $a_n > l - 0.1l = 0.9l$ whenever $n \geq N_1$. That means that $\frac{|a_n - l|}{|a_n l|} \leq \frac{1}{0.9l}\,|a_n - l|$ for all $N \geq N_1$. Next, given ε_2 (to be specified later), let N_2 be such that $|a_n - l| \leq \varepsilon_2$ for all $n \geq N_2$. Then for $n \geq \max(N_1, N_2)$ we have
$$\left|\frac{1}{a_n} - \frac{1}{l}\right| \leq \frac{1}{0.9l}\,|a_n - l| \leq \frac{\varepsilon_2}{0.9l}$$
   So given ε, choose ε_2 such that $\varepsilon = \frac{\varepsilon_2}{0.9l}$ (i.e. $\varepsilon_2 = 0.9l \times \varepsilon$), and choose $N \geq \max(N_1, N_2)$. Then $\left|\frac{1}{a_n} - \frac{1}{l}\right| < \varepsilon$ for all $n \geq N$. ■

4. [BONUS] The *logistic map* is defined iteratively by $x_{n+1} = r x_n (1 - x_n)$ where r is a parameter. Take $x_0 = 0.5$ for simplicity.

   (b) Fix $r = 1.5$, and compute the first 10 iterates of x_n. You can use a computer for this. What is the limit $\lim_{n\to\infty} x_n$ for this value of r?

   (c) Repeat part (b) for $r = 2,\ 2.5,\ 3,\ 3.2,\ 3.5,\ 3.52$. Comment on anything interesting that you observe.

   (d) You should see that something weird happens at $r = 3$. For $r \in (1, 3)$, based on your observations from part (c), can you conjecture what is the limit $\lim_{n\to\infty} x_n$ as a function of r?

   (e) Conjecture what happens when $r \in (3, 3.5)$.

5. [BONUS]: Consider a sequence $x_{n+1} = x_n^{x_n}$, with $x_0 = 0.5$. Using a computer, convince yourself that $x_n \to 1$ *very slowly.* Using a computer, determine how big should n be, so that $|x_n - 1| < 0.01$? so that $|x_n - 1| < 0.001$? Based on your observations, guess how big should n be so that $|x_n - 1| < \varepsilon$ for any given (small) ε?

6. Let $f(x) = \frac{1}{2-x}$.

   (a) Find a number $\delta > 0$ such that $|f(x) - f(1)| \leq 0.1$ whenever $|x-1| < \delta$.

   (b) Given an $\varepsilon > 0$, find a number $\delta > 0$ such that $|f(x) - f(1)| \leq \varepsilon$ whenever $|x-1| < \delta$.

   (c) Conclude that $f(x)$ is continuous at $x = 1$.

   Solution. (a) $f(1) = 1$. So we want

$$\left|\frac{1}{2-x} - 1\right| \leq 0.1 \iff$$

$$\begin{aligned} 0.9 \leq \frac{1}{2-x} &\leq 1.1 \iff \\ 1/1.1 \leq 2-x &\leq 1/0.9 \iff \\ 0.8889 \leq x &\leq 1.0909 \\ -0.1111 \leq x-1 &\leq 0.0909. \end{aligned}$$

   So choose $\delta = 0.0909$ (or any smaller will also do). Then run these inequalities backwards, showing that $\left|\frac{1}{2-x} - 1\right| \leq 0.1$ whenever $|x-1| < \delta$.

   (b) Similar to part (a) we have:

$$\begin{aligned} \left|\frac{1}{2-x} - 1\right| &\leq \varepsilon \iff && (1) \\ \frac{1}{1+\varepsilon} \leq 2-x &\leq \frac{1}{1-\varepsilon} \iff \\ \frac{-\varepsilon}{1+\varepsilon} \leq 1-x &\leq \frac{\varepsilon}{1-\varepsilon} && (2) \end{aligned}$$

   Equation (2) holds when $|1-x| \leq \min\left(\frac{\varepsilon}{1+\varepsilon}, \frac{\varepsilon}{1-\varepsilon}\right) = \frac{\varepsilon}{1+\varepsilon}$. So we choose $\delta = \frac{\varepsilon}{1+\varepsilon}$.

   (c) Part (b) shows that $\lim_{x\to 1} f(x) = f(1)$. Hence $f(x)$ is continuous at $x = 1$.

7. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions on the interval $[-4, 4]$.

   Solution. Let $f(x) = x^3 - 15x + 1$. Then $f(-4) < 0,\ f(0) > 1,\ f(1) < 0$ and $f(4) > 0$. Hence by intermediate value theorem, there is a root in $(-4, 0)$, another root in $(0, 1)$ and a third root in $(1, 4)$.

8. Show that the function $f(x) = \sin(x-a)\sin(x-b) + x$ has the value $(a+b)/2$ at some point x.

   Solution. Note that $f(a) = a$ and $f(b) = b$. So there exists c such that $f(c) = (a+b)/2$.

9. Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Show that there is a number $c \in [0, 1]$ such that $f(c) = c$.

   Solution. Let $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Hence there is a $c \in [0, 1]$ such that $g(c) = 0$ or $f(c) = c$.

10. Prove that at any instant in time, there exist two points on the equator that have the same temperature and that are antipodal to each other. (hint: you may assume that the temperature is a continuous function of space).

    Solution. Let $T(x)$ be the temperature on the equator, where x is in degrees of longitude (from -180 to 180). Let $g(x) = T(x) - T(x+180)$. Then $g(-180) = -g(0)$ since $T(-180) - T(180)$. So by intermediate value theorem, there is an x such that $g(x) = 0$, or $T(x) = T(x+180)$.