Homework 3

Recall the definition of a limit $\lim f(x)$

1. Using the rigorous definition, prove the following version of the Squeeze theorem: if $0 \leq f(x) \leq g(x)$ and $g(x) \to 0$ as $x \to a$, then $f(x) \to 0$ as $x \to a$.

   [Recall that we say that $f(x) \to l$ as $x \to a$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - l| \leq \varepsilon$ whenever $0 < |x - a| < \delta$].

2. (a) Using the definition, compute the derivative of $f(x) = \frac{1}{2+3x^2}$. (b) Verify your answer by using differentiation rules.

3. Find equations of two straight lines that are tangent to $y = \frac{x^2}{x-1}$ and pass through a point $(2, 0)$.

4. Find derivative of

   (a) $y = \sqrt{5x^2 + 3}$

   (b) $y = \cos(2\sin(3x^4))$

   (c) $y = \left(\sqrt{x} - 3x^3\right) x^{-5} \; + \cos(3)$

   (d) $y = \left(\frac{2x-1}{3x+1}\right)^4$

5. Suppose that $f(x) = \left(\frac{1}{g(x)}\right)^2$. Compute $f'(x)$ and $f''(x)$ in terms of $g(x)$ and its derivatives.

6. Show that $\frac{d}{d\theta} \cot\theta = -\csc^2(\theta)$. [you can use differentiation rules and things like $\sin'(\theta) = \cos(\theta)$ etc].

7. Suppose that $f(0) = 0$ and $f'(x) \geq 1$ for all x. What can you say about $f(2)$? [Hint: use mean value theorem].

8. (a) Suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

   (b) Suppose that $f(0), f'(0), \ldots, f^{(n)}(0) = 0$ and $f^{(n+1)}(x) \geq 0$ for all $x \geq 0$. Show that $f(x) \geq 0$ for all $x \geq 0$.

9. (a) Prove that $\sin x \leq x$ for all $x \geq 0$.

   (b) You are given a function $f(x)$ with the following properties: $f'(x) = \frac{\sin x}{x}$; $\quad f(0) = 0$. Show that $f(\pi) \leq \pi$.

10. (a) Show that $\sin(x) \geq x - \frac{x^3}{6}$ for all $x \geq 0$ [Hint: use q8 part b].

    (b) You are given a function $f(x)$ with the following properties: $f'(x) = \frac{\sin x}{x}$; $\quad f(0) = 0$. Find a number A such that $f(\pi) > A$.