Homework 3

Recall the definition of a limit $\lim f(x)$

1. Using the rigorous definition, prove the following version of the Squeeze theorem: if $0 \leq f(x) \leq g(x)$ and $g(x) \to 0$ as $x \to a$, then $f(x) \to 0$ as $x \to a$.

   [Recall that we say that $f(x) \to l$ as $x \to a$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - l| \leq \varepsilon$ whenever $0 < |x - a| < \delta$].

   Solution. Since $g \to 0$ as $x \to a$, given an ε, there exists δ such that $|g(x)| \leq \varepsilon$ whenever $0 < |x - a| \leq \delta$. But since $0 \leq f(x) \leq g(x)$, it follows that $|f(x)| \leq |g(x)| \leq \varepsilon$. So the same delta that works for g also works for f! In other words, $|f(x) - 0| \leq \varepsilon$ whenever $|x - a| < \delta$, for the delta as chosen above.

2. (a) Using the definition, compute the derivative of $f(x) = \frac{1}{2+3x^2}$. (b) Verify your answer by using differentiation rules.

   Solution. (a)

$$
\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\frac{1}{2+3(x+h)^2} - \frac{1}{2+3x^2}}{h} \\
&= \frac{3x^2 - 3(x+h)^2}{h(2+3x^2)(2+3(x+h)^2)} \\
&= \frac{-6xh + O(h^2)}{h(2+3x^2)(2+3(x+h)^2)} \\
&\to -\frac{6x}{(2+3x^2)^2}
\end{aligned}
$$

   (b) Yep!

3. Find equations of two straight lines that are tangent to $y = \frac{x^2}{x-1}$ and pass through a point $(2, 0)$.

   Solution. We have $y' = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$. If a line passes through (x_0, y_0) and $(2, 0)$ and is tangent to that curve, we must then have

$$
\frac{y_0 - 0}{x_0 - 2} = \frac{x_0^2 - 2x_0}{(x_0 - 1)^2} \text{ where } y = \frac{x_0^2}{x_0 - 1}.
$$

   Solving this, we obtain either $x_0 = 0$ or $x_0 = \frac{4}{3}$. Moreover, $y'(0) = 0,\ y'(\frac{4}{3}) = -8$. So the two lines are:

$$
\begin{aligned}
y &= 0 \\
y - 0 &= -8(x - 2).
\end{aligned}
$$

4. Find derivative of

   (a) $y = \sqrt{5x^2 + 3}$

   (b) $y = \cos(2\sin(3x^4))$

   (c) $y = \left(\sqrt{x} - 3x^3\right)x^{-5} \ + \cos(3)$

   (d) $y = \left(\frac{2x-1}{3x+1}\right)^4$

   Solution. (a) $\left(5x^2 + 3\right)^{-1/2}(5x)$, (b) $-24\sin(2\sin(3x^4))\cos(3x^4)x^3$, (c) expand $y = x^{-4.5} - 3x^{-2} + \cos 3$ so that $y' = -4.5x^{-5.5} + 6x^{-3}$. (d) $4\left(\frac{2x-1}{3x+1}\right)^3\left(\frac{2(3x+1)-3(2x-1)}{(3x+1)^2}\right) = 20\frac{(2x-1)^3}{(3x+1)^5}$.

5. Suppose that $f(x) = \left(\frac{1}{g(x)}\right)^2$. Compute $f'(x)$ and $f''(x)$ in terms of $g(x)$ and its derivatives.

   Solution. $f = g^{-2}, f' = -2g^{-3}g', \quad f'' = 6g^{-4}g'^2 - 2g^{-3}g'' = g^{-4}\left(6g'^2 - 2gg''\right)$.

6. Show that $\frac{d}{d\theta}\cot\theta = -\csc^2(\theta)$. [you can use differentiation rules and things like $\sin'(\theta) = \cos(\theta)$ etc].

   Solution.
$$\begin{aligned}\frac{d}{d\theta}\cot\theta = \frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) &= \\ &= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} = \frac{-1}{\sin^2\theta} = -\csc^2(\theta).\end{aligned}$$

7. Suppose that $f(0) = 0$ and $f'(x) \geq 1$ for all x. What can you say about $f(2)$? [Hint: use mean value theorem].

   Solution. The "extreme" case if $f'(x) = 1$. Then $f(x) = x$ so that $f(2) = 2$. In general, $f(x) > 2$. Because $(f(2) - f(0))/2 = f'(x) \geq 1 \implies f(2) \geq 2$.

8. (a) Suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

   (b) Suppose that $f(0), f'(0), \ldots, f^{(n)}(0) = 0$ and $f^{(n+1)}(x) \geq 0$ for all $x \geq 0$. Show that $f(x) \geq 0$ for all $x \geq 0$.

   Solution. (a) Let $h(x) = f(x) - g(x)$. Then $h'(x) \leq 0$ for all $x \geq 0$ and $h(0) = 0$. So h is decreasing and starts at zero, so it keeps below zero.

   (b) $h(x) = f(x) - g(x)$.Then $h^{(n+1)}(x) \leq 0$ for all $x \geq 0$ and $h^{(n)}(0) = 0$.So $h^{(n)}$ is decreasing and starts at zero; hence $h^{(n)}(x) \leq 0$ for all $x \geq 0$. Now repeat this argument $n-1$ times to show that $h(x) \leq 0$ for all $x \geq 0$.

9. (a) Prove that $\sin x \leq x$ for all $x \geq 0$.

   (b) You are given a function $f(x)$ with the following properties:$f'(x) = \frac{\sin x}{x}; \quad f(0) = 0$. Show that $f(\pi) \leq \pi$.

   Solution. (a) Let $h(x) = \sin x - x$. Then $h'(x) = \cos x - 1 \leq 0$ and $h(0) = 0$. So h is negative for positive x.

   (b) By part (a), we know that $f'(x) \leq 1$. Then apply the mean value theorem: $(f(\pi) - f(0))/\pi = f'(x) \leq 1$. So $f(\pi) \leq \pi$.

10. (a) Show that $\sin(x) \geq x - \frac{x^3}{6}$ for all $x \geq 0$ [Hint: use q8 part b].

    (b) You are given a function $f(x)$ with the following properties:$f'(x) = \frac{\sin x}{x}; \quad f(0) = 0$. Find a number A such that $f(\pi) > A$.

    Solution. (a) Let $h(x) = \sin x - x + \frac{x^3}{6}$. We have:
$$\begin{aligned}h'(x) &= \cos x - 1 + \frac{x^2}{2} \\ h''(x) &= -\sin x + x \\ h'''(x) &= -\cos x + 1\end{aligned}$$

    So we have:
$$\begin{aligned}h(0) = h'(0) &= h''(0) = 0 \\ h'''(x) &\geq 0\end{aligned}$$

    It then follows from q8 that $h(x) \geq 0$ for $x \geq 0$, which is what we needed to show.

    (b) From part (a) $\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}$. Then $f'(x) \geq 1 - \frac{x^2}{6}$ and $f(0) = 0$. So then $f(x) \geq x - \frac{x^3}{18}$ and in particular $f(\pi) \geq \pi - \frac{\pi^3}{18} \geq 0.419$.