MATH 1500, Homework 7

Not to be handed in.

1. Sketch the following function (without using computer):

$$y = \frac{(x-1)(x-2)}{(x-3)^2}$$

   Indicate any max/min, or inflection points.

2. Go through examples of Section 4.6.
3. Section 4.6, problems 1-8 (do a selection)
4. Go through examples in section 4.8, including this one:

EXAMPLE 5 A man can run twice as fast as he can swim. He is standing at point A on the edge of a circular swimming pool 40 m in diameter, and he wishes to get to the diametrically opposite point B as quickly as possible. He can run around the edge to point C, then swim directly from C to B. Where should C be chosen to minimize the total time taken to get from A to B?

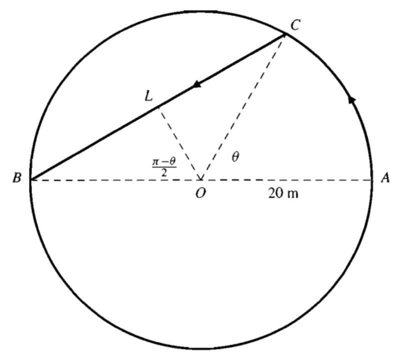


Figure 4.55 Running and swimming to get from A to B

Solution It is convenient to describe the position of C in terms of the angle AOC, where O is the centre of the pool. (See Figure 4.55.) Let θ denote this angle. Clearly $0 \le \theta \le \pi$. (If $\theta = 0$, the man swims the whole way; if $\theta = \pi$, he runs the whole way.) The radius of the pool is 20 m, so arc $AC = 20\theta$. Since angle $BOC = \pi - \theta$, we have angle $BOL = (\pi - \theta)/2$ and chord $BC = 2BL = 40\sin((\pi - \theta)/2)$.

Suppose the man swims at a rate k m/s and therefore runs at a rate $2k$ m/s. If t is

5. Do a selection of problems from 4.8, including (but not limited to) 6, 11, 13, 24.
6. Use linear approximation to estimate a given number (without using a calculator).
   (a) $\frac{1}{2.01}$ (b) $\ln(0.5)$ (expand around $x = 1$). (c) $9^{1/3}$.
7. (a) Use linear approximation with error bounds to show that

$$x - \frac{x^2}{2} < \ln(1+x) < x \text{ when } x > 0.$$

   (b) Conclude that

$$e^{1-x/2} < (1+x)^{1/x} < e \qquad (1)$$

   (c) Use that fact that $e \le 4$ (which we showed in class) and (1) to prove that $e \le 3$.

8. A function $f(x)$ has the following properties:

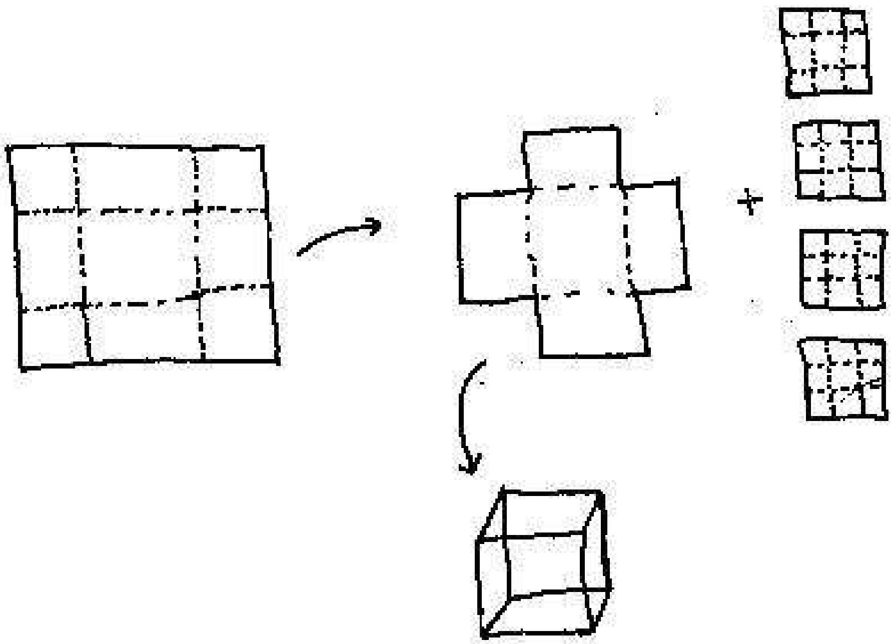
$$f(0) = 1, f'(0) = 2, \;\; f''(x) = x^2 + \sin(30e^x) - 0.01x^{1/2}.$$

   Estimate $f(0.1)$ and give error bounds.

9. (a) Use linear approximation to estimate the root of $e^x(x+1) = 1.1$ that is close to $x = 0$.

   (b) Use Newton's method to estimate x to 10 decimal places.

10. [BONUS: hand in for marks] Take a square sheet of paper whose side is of length l. From each of the four corners, cut out a smaller square each of whose length is a fraction r_1 of l, with $0 < r_1 < 1/2$. From the resulting cross, make a box of height lr_1 and of base length $l\,(1-2r_1)$. For each of the remaining four squares, cut out four corners, each of whose length is a fraction r_2 of the length of the small square. Then make four additional boxes from the four resulting crosses.



   (a) How should r_1 and r_2 be chosen in order to maximize the total volume of the resulting five boxes?

   (b) Continue the procedure indefinitely, defining a sequence of ratios $r_1, r_2, r_3 \ldots$ and resulting in $1, 5, 21, \ldots$ boxes. Suppose that it is required that all these ratios are the same: $r = r_1 = r_2 = r_3 = \ldots$. How should you choose r in order to maximize the total volume?

   (c) Now suppose that you are free to choose $r_1, r_2, r_3 \ldots$ independently from one another, in such a way as to maximize the total volume. Would you get a different value for r_1 than what you found for r in part b? If the same, why? If different, what would it be?