Math 1500, Midterm 1

No calculator or cheat sheets are allowed. Please write all your answers in the booklet provided.

1. Find the following limits (answer can be either a number or $\pm\infty$)

   (a) $\displaystyle\lim_{x\to\infty} \frac{x^2 + \sqrt{2 + 3x + 4x^4}}{4x + 5x^2}$ (b) $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{x}$ (c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

2. (a) State the delta-epsilon definition of a limit: $\lim_{x\to a} f(x) = l$. (b) Find a δ such that $\left|\frac{1}{x} - \frac{1}{2}\right| \le 0.1$ whenever $|x - 2| \le \delta$. (c) Use the delta-epsilon definition of a limit to show that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

3. (a) Write down the definition of derivative of $f(x)$ as a limit. (b) Let $f(x) = 1/x$. Use the definition of derivative as a limit to directly compute $f'(2)$. (No marks will be given for using differentiation rules).

4. (a) State the mean value theorem. (b) Use the mean value theorem to show that if $f'(x) \ge 0$ on $[a, b]$ then $f(x)$ is increasing (i.e. nondecreasing) on $[a, b]$.

5. (a) When a snowball is melting, its volume is decreasing at a rate proportional to its surface area. How fast is the radius of the snowball decreasing? Note: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$ where V is volume, S is surface area and r is the radius of a sphere. (b) It was found that the radius r decreased from 5 cm to 4 cm in 2 minutes. How long will it take for the snowball to melt completely?

1) (a) $\frac{3}{5}$

(b) $\frac{\sin(2x)}{x} \sim \frac{2x}{x} \to 2$ as $x \to 0$

(c) $\frac{\sqrt{2+x} - \sqrt{2}}{x} = \frac{\left(\sqrt{2+x} - \sqrt{2}\right)\left(\sqrt{2+x} + \sqrt{2}\right)}{x\left(\sqrt{2+x} + \sqrt{2}\right)}$

$= \frac{x}{x\left(\sqrt{2+x} + \sqrt{2}\right)} \to \frac{1}{2\sqrt{2}}$ as $x \to 0$

2) (a) $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|f(x) - l| \leq \varepsilon$ whenever $0 < |x-a| \leq \delta$.

(b) $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|2-x|}{|2x|} = \frac{|2-x|}{2x} \leq 0.1$ [assuming $x > 0$]

want this.

If $|2-x| < \delta$ then $x > 2 - \delta \Rightarrow \frac{1}{x} < \frac{1}{2-\delta}$

$\Rightarrow \frac{|2-x|}{2x} \leq \frac{\delta}{4-2\delta} \leq 0.1$

want this

So choose $\frac{\delta}{4-2\delta} = 0.1 \Leftrightarrow \delta = \frac{0.1 \times 4}{1 + 0.1 \times 2} = 1/3$

$\boxed{\delta = 1/3}$

(c) Repeating (b) but replacing "0.1" by "ε" we get $\boxed{\delta = \frac{4\varepsilon}{1+2\varepsilon}}$.

3) (a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(b) $\dfrac{\frac{1}{2+h} - \frac{1}{2}}{h} = \dfrac{-h}{(2+h)2h} \longrightarrow -\dfrac{1}{2^2} = \dfrac{-1}{4}$ as $h \to 0$

4) (a) $\exists\ c \in [a,b]$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b-a}$ [provided that f' exists in $[a,b]$

(b) Given $x_1 < x_2$, we have $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0$

$\Rightarrow f(x_2) - f(x_1) \geq 0$

$\Rightarrow f(x_2) \geq f(x_1)$

And this is true for <u>any</u> $x_1 < x_2 \in [a,b]$.

5) (a) $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt} = -cS \Rightarrow \boxed{\dfrac{dr}{dt} = -c}$ $4\pi r^2$

(b) From (a), $r = -ct + c_2$ for some c, c_2.

Moreover, $r(0) = 5$, $r(2) = 4 \Rightarrow \begin{cases} 5 = c_2 \\ 4 = -c2 + c_2 \Rightarrow c_1 = \frac{1}{2} \end{cases}$

$\Rightarrow \boxed{r(t) = 5 - \frac{1}{2}t}$ setting $r = 0 \Rightarrow \boxed{t = 10}$