Math1500, Midterm 2

1. Find the limit
$$\lim_{t\to 0}\frac{\cos t-1}{\ln(1+t^2)}$$

2. Compute the first three non-zero terms of Taylor series expansion of
$$y=\frac{\cos(x)}{1-x}.$$
near $x=0$.

3. The function $f(x)$ has the following properties: $f(0)=0, f'(0)=1, f''(0)=2$ and $f'''(x)=\frac{\sin x}{4-x^2}$. Estimate $f(1)$ as best as you can, and give the tightest error bound that you can.

4. Evaluate the following integrals.
$$(a)\ \int\frac{e^x dx}{1+e^{2x}}$$
$$(b)\ \int \arctan(x)\ dx$$
$$(c)\ \int_0^{1/2}\sqrt{x-x^2}$$

5. (a) Write *the form* of Partial Fraction Decomposition for
$$f(x)=\frac{1}{(x^2-x)\,(x^2+1)^2}$$
Note: do **not** to solve for constants.

(b) Express $\int f(x)dx$ in terms of various constants that appear in part (a) (you do not need to solve for these constants).

6. [BONUS]: Suppose that $f(x)$ is a differentiable function. Show that $\int_0^\pi \sin(nx)\,f(x)dx\to 0$ as $n\to\infty$.

1) $\cos t \sim 1 - \frac{t^2}{2} + \cdots$; $\ln(1+t^2) = t^2 + \cdots$

$$\Rightarrow \frac{\cos t - 1}{\ln(1+t^2)} \sim \frac{-t^2/2}{t^2} \sim -\frac{1}{2} \text{ as } t \to 0$$

2) $\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{120} + \cdots$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$$\Rightarrow \frac{\cos x}{1-x} \sim \left(1 - \frac{x^2}{2} \cdots\right)\left(1 + x + x^2 \cdots\right)$$

$$\sim \boxed{1 + x + \frac{x^2}{2}} + \cdots$$

3) $f(1) = 1 + 1 + E$ where

$$E = \frac{\sin t}{4 - t^2} \frac{1}{3!} (1)^3 \quad , \quad t \in [0, 1].$$

Now $0 \leq \sin t \leq \sin 1$;

$\frac{1}{4} \leq \frac{1}{4-t^2} \leq \frac{1}{3}$

So that $\boxed{0 \leq E \leq \frac{\sin 1}{3} \frac{1}{6}} \leq \frac{1}{18}$

Conclusion : $\boxed{0 \leq f - 2 \leq \frac{\sin 1}{18}}$

4)

a) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx \qquad \left(u = e^x\right) \quad du = e^x\,dx$

$\displaystyle= \int \frac{du}{1+u^2} = \arctan u = \arctan(e^x)$

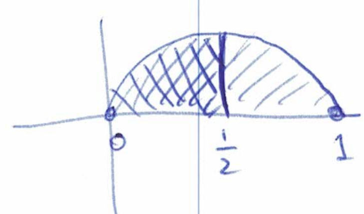
b) $\displaystyle\int \arctan x\,dx = (\arctan x)\,x - \underbrace{\int \frac{x}{1+x^2}\,dx}_{\frac{\ln(1+x^2)}{2}}$

$du = dx \qquad v = \arctan x$

$u = x \qquad dv = \dfrac{1}{1+x^2}\,dx$

$\displaystyle= (\arctan x)\,x - \frac{\ln(1+x^2)}{2}$

c) $\displaystyle\int_0^{\frac{1}{2}} \sqrt{x-x^2}\,dx =$



$= \left(\frac{1}{2}\right)^2 \frac{\pi}{4}$

$= \dfrac{\pi}{16}$

Complete square:

$x^2 - x = \left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 ; \quad u = x - \frac{1}{2}$

$x = 0 \Rightarrow u = -\frac{1}{2}$

$x = \frac{1}{2} \Rightarrow u = 0$

$\displaystyle I = \int_0^{-\frac{1}{2}} \sqrt{\left(\frac{1}{2}\right)^2 - u^2}\,du$

$u = \dfrac{\sin\theta}{2}$

$u = 0 \Rightarrow \theta = 0$

$u = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{2}$

$\displaystyle\Rightarrow I = \frac{1}{4}\int_0^{-\frac{\pi}{2}} \cos^2\theta = \frac{\pi}{16}$

5) $f = \dfrac{1}{x(x-1)(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C+Dx}{x^2+1} + \dfrac{E+Fx}{(x^2+1)^2}$

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$\int f = A\ln x + B\ln(x-1) + C\arctan x$

1

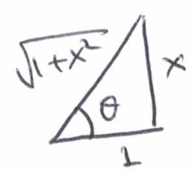
$+\dfrac{D}{2}\ln(x^2+1) + E\underbrace{\int\dfrac{1\,dx}{(x^2+1)^2}}_{I_1} + F\underbrace{\int\dfrac{x\,dx}{(x^2+1)^2}}_{I_2}$

2 $\underline{I_2}$: $u = x^2+1$; $du = 2x\,dx$; $I_2 = \dfrac{1}{2}\int\dfrac{du}{u^2} = -\dfrac{1}{2}\dfrac{1}{u} = -\dfrac{1}{2}\dfrac{1}{(x^2+1)}$

$\underline{I_1}$: $\int\dfrac{1}{(1+x^2)^2}\,dx$ $\boxed{x = \tan\theta}$

2

$dx = \sec^2\theta\,d\theta$



$= \int\dfrac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = \int\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\,d\theta$

$= \dfrac{1}{2}\theta + \dfrac{1}{4}\underbrace{\sin 2\theta}_{2\underbrace{\sin\theta}_{\frac{x}{\sqrt{1+x^2}}}\underbrace{\cos\theta}_{\frac{1}{\sqrt{1+x^2}}}}$

$= \dfrac{1}{2}\arctan x + \dfrac{x}{1+x^2}$

$$6)\ I = \int \underbrace{\sin(nx)}_{\frac{d}{dx}\frac{(-\cos(nx))}{n}} f(x) = \underbrace{\frac{-\cos(nx)}{n} f(x)\Big|_0^{\pi}}_{A} + \underbrace{\frac{1}{n}\int \cos(nx) f(x)\,dx}_{B}$$

<u>Now</u> $\quad A = \frac{1}{n}\left(-f(0) \pm f(\pi)\right) \to 0$ as $n \to 0$

<u>and</u> $\quad |B| \le \frac{1}{n}\int |\cos(nx)\, f(x)|\,dx$

$$\le \frac{1}{n}\int |f(x)|\,dx \longrightarrow 0$$

as $n \to 0$

$\Rightarrow \quad I \to 0$