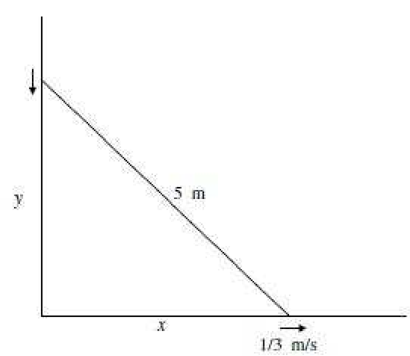
Math 1500, Final Exam

No calculator or cheat sheets are allowed. You have 2.5 hours. Please write all your answers in the booklet provided.

1. [6 points]
   (a) Find $\lim_{x\to 0} \frac{x^2}{(\sin(2x))^2}$
   (b) Find $\lim_{x\to\infty} \frac{x^2+\sin x}{\sqrt{4x^4+\cos x}}$.
2. [6 points]
   (a) State the delta-epsilon definition of continuity of $f(x)$ at $x=a$.
   (b) Show that the function $f(x)=1/x$ is continuous at $x=1$.
3. [8 points]
   (a) Find the derivative of $f(x)=\sqrt{x}$ from the first principles (no credit given for using differentiation rules).
   (b) Find the derivative of $y=\cos\left(\frac{x^2}{\sqrt{x}}\right)$ using any method you like.
   (c) Find the derivative of $y=(\sin x)^{\cos x}$ using any method you like.
   (d) Find the derivative of $f(x)=\int_3^{\ln(x)} \tan t\, dt$.
4. [5 points] Sketch the graph of $y=x-\frac{1}{x^2}$. Include any max/min, roots, asymptotes.
5. [6 points]
   (a) Estimate $\ln(1.1)$ using linear approximation about an appropriate point.
   (b) Estimate the error in part (a).
6. [6 points]
   (a) Determine the general form of the Taylor series expansion of $f(x)=\frac{\sin(x)}{x}$ about $x=0$.
   (b) Let $F(x)=\int_0^x \frac{\sin(t)}{t}dt$. Determine the general form of the Taylor series expansion of $F(x)$ and estimate $F(0.5)$ using the first two terms.
7. [6 points] A ladder 5 meters long is placed against a wall. The ladder begins to slip. How fast is the top of the ladder sliding down when the bottom of the ladder is 3 meters away from the wall and is moving to the right at 1/3 meters/sec?



8. [8 points] A person in a row boat at point P is 4km away from a straight shore line. The point A on the shore is directly opposite the boat. The objective is to travel from point P to point B on the shore a distance 9km from A in a minimum amount of time. If the person can row at 4 km per hour and walk at 5 km per hour, where should the person land the boat between A and B? Verify that the answer you obtained is indeed a minimum.

9. [8 points] A corpse was discovered in a motel room and its temperature was 26°C. The temperature of the room is kept constant at 15°C. Two hours later the temperature of the corpse dropped to 24°C. The normal human temperature is 37°C. Assuming the person was healthy at the time of murder, how long was he dead before he was found? You can leave your answer in terms of logs. You may assume Newton's law of cooling; that is the temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of the surrounding environment.

10. [6 points]

    (a) Evaluate $\int x^2 \exp(x^3+1)dx$

    (b) Determine the area of the region bounded by the curve $y = 1/x$ and a line that goes through $(1/2, 2)$ and $(2, 1/2)$.

    (c) Show that $\int_1^2 \sqrt{x^4-1}dx \leq \frac{7}{3}$. Hint: $x^4 - 1 \leq ???$

11. [BONUS, 6 points]

    (a) Suppose that $f(0) = -1$ and $f''(x) > 0$ for all $x \geq 0$. Show that $f(x)$ has at most one positive root.

    (b) Sketch an example of a function $f(x)$ with that satisfies the conditions of part (a) but has no positive roots.

    (c) Suppose that $f(0) = -1$ and $f''(x) \geq 1$ for all $x \geq 0$. Show that $f(x)$ has exactly one positive root.