MATH 1500, Homework 13

Due date: Wed, 10 February

1. (a) Prove that the equation $x = \cos x$ has precisely one solution.
   (b) Find its solution using an iteration $x_{n+1} = \cos(x_n)$. starting with $x_1 = 1$. How many iterations were required to get two decimal digits? [Make sure your calculator/computer is set to "radians" mode].
   (c) Prove that the iteration that you used in part (b) will eventually converge.
   (d) Use Newton's method to find the root of $x = \cos x$. List the first five iterations starting with $x = 1$. How many iterations were required to get the first 8 digits?

2. Consider the equation

$$x = A\cos x. \tag{1}$$

In Question 1, you showed that the the solution is unique if $A = 1$. However for larger values of A, this equation can have more than one solution (see the graph below).

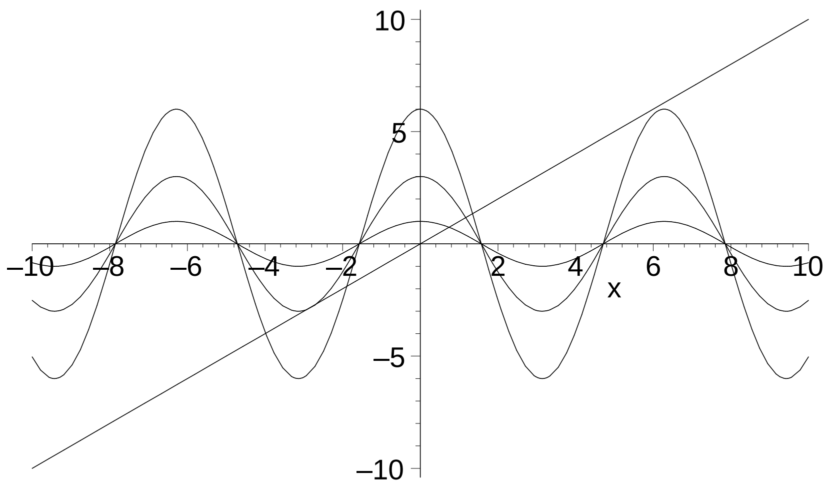

Figure 1: Graph of $y = \cos x$, $y = 3\cos x$, $y = 6\cos x$ and $y = x$.

   (a) Note from the graph that a new solution to (1) appears around $A = 3$. So there exists a smallest A, call it A_1 so that (1) has one solution when $A < A_1$, has two solutions when $A = A_1$ and has three (or more) solutions when $A > A_1$. Formulate a set of equations that A_1 satisfies.
   (b) Apply Newton's method to find A_1 to 4 decimal places.
   (c) Similar to part (a), there exists A_2 such that (1) has three solutions when $A < A_3$ but has (at least) five if $A > A_2$. Find A_2 to 4 decimal places using Newton's method.

3. Let

$$f(x) = x + a(e^{-x} - 1)$$

and consider the map $x_{n+1} = f(x_n)$.

   (a) Show that $x = 0$ is a fixed point of this map.
   (b) For which values of a is this fixed point stable?

(c) [Bonus] Use a computer (e.g. Matlab, Maple, Java, C or any other programming language) to sketch a bifurcation diagram of $f(x)$ with $a \in [1.8, 4]$. That is, for values of a between 1.8 and 4 with increments of (say) 0.01, plot x_n for (say) $n = 100$ to 1100.

4. As we saw in class, Newton's method gives an excellent convergence in most cases. However the convergence can be relatively poor if f has a double root i.e. $f(r) = f'(r) = 0$.

   (a) Illustrate this by using Newton's method to find the root of $f(x) = x^2$. What kind of convergence do we have in this case?

   (b) [Bonus] Generalize part (a) to a general function $f(x)$ with $f(r) = 0,\ f'(r) = 0$ and $f''(r) \neq 0$. Show that if $E_n = |x_n - r|$ then $E_{n+1} \sim \frac{1}{2} E_n$ when x_n is sufficiently close to r.