MATH 1500, Homework 14

Due date: Wed, 17 February

1. Consider $I = \int_0^1 \sqrt{1+x^2}dx$.

   (a) Estimate I using the Trapezoid rule with $n = 4$ subintervals.

   (b) Estimate I using the Simpson's rule with $n = 4$ subintervals.

   (c) How many subintervals n are needed to estimate I with Midpoint rule to within 10^{-5}? Note that $|I - M_n| \le \frac{Mh^2(b-a)}{24}$ where $M \ge \max_{[a,b]} |f''(x)|$ and $h = \frac{b-a}{n}$.

   (d) Use Romberg integration to compute I numerically to within 10^{-5} (Use enough iterations until you see that the answer doesnt change anymore in the first 5 digits).

2. Consider $I = \int_0^1 4\sqrt{1-x^2}dx$.

   (a) Verify that $I = \pi$.

   (b) Evaluate I numerically using the midpoint rule M_n with $n = 1, 2, 4$. Then compute the error $E_n = \pi - M_n$. Note: you may use $\pi = 3.141592654$.

   (c) Repeat part (b) but using Simpson's rule.

   (d) From class, we know that Midpoint rule has $O(h^2)$ error whereas Simpson's rule has $O(h^4)$ error if the integrand is sufficiently smooth. Is this the case for this integral? Why not? Use computations of parts (b) and (c) in support of your answer.

   (e) Rewrite the integral I by making a substitution $\sqrt{1-x} = t$. Then apply Simpson's rule with $n = 2$ and $n = 4$. Comment on the error behaviour. Why is it better than part (c)?

   (f) [Bonus] Use Romberg integration and part (e) verify the first 10 digits of π. How many subintervals did you need?

3. The goal of this question is to estimate the error for the Midpoint rule.

   (a) Consider $\int_{x_0}^{x_1} f(x)dx$ with $x_1 = x_0 + h$; and where f has a bounded second derivative. Recall the Taylor remainder theorem with $N = 1$ states that

   $$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(\xi) \text{ where } \xi = \xi(x) \in [a, x].$$

   Apply this expansion with $a = x_0 + h/2$ to estimate $\int_{x_0}^{x_1} f(x)dx - hf(x_0 + h/2)$.

   (b) Prove the error formula for the midpoint rule: if M_n is the midpoint rule with $n = (b-a)/h$ subintervals, then $\left|\int_a^b f(x) - M_n\right| \le \frac{M}{24}(b-a)h^2$ where $M \ge \max_{[a,b]} |f''(x)|$.

4. [Bonus] Show that the error in using the right-point rule for $\int_0^1 x^{-1/2}$ is of the order $O\left(h^{1/2}\right)$.