MATH 1500, Homework 17

Due date: 31 March (Wed)

1. (a) Let $f(x) = |x|$. Determine the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

for $f(x)$.

(b) Evaluate the above identity at $x = 1$ to arrive at a formula for π in terms of an infinite sequence.

(c) Given a continuous function $f(x)$ whose Fourier expansion is given by

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

Show that

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx = \pi \sum_{n=0}^{\infty} a_n^2.$$

(d) Apply (c) to the function $f(x) = |x|$ to arrive at the formula for π^2.

2. (a) Using the identity $\cos t = \frac{e^{it}+e^{-it}}{2}$, show that

$$\cos^3 \theta = \frac{1}{4} (\cos 3\theta + 3\cos\theta).$$

Hint: recall that $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$.

(b) Derive a similar identity for $\sin^3(\theta) = ...$

(c) Use $\cos t = \frac{e^{it}+e^{-it}}{2}$ and geometric series to show that

$$1 + 2\sum_{n=1}^{N} \cos(nx) = \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{\sin(x/2)}$$

3. Determine the real and imaginary part of $\frac{e^{1+2i}}{3+4i}$.

4. Determine all (complex) roots of $z^3 = -8$. Sketch them on a graph.

5. Consider the ODE

$$y'' - 2y' + 5y = 0. \tag{1}$$

(a) Find the most general solution to this ODE.

(b) Solve the ode (1) subject to the initial conditions $y(0) = 1, y'(0) = -1$.

(c) Find the solution to the inital value problem

$$\begin{cases} y'' - 2y' + 5y = e^x + x. \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

6. Consider the following problem:

$$y'' + \mu y' + y = \cos(\Omega x).$$

(a) Find a particular solution of the form $y = A\cos(\Omega x) + B \sin \Omega x$.

(b) [Bonus] Plot the graph of of B as a function of Ω with $\mu = 2, 1, 0.5, 0.1$. Where is the maximumum of B? What happens to it as $\mu \to 0$?

(c) [Bonus] For a fixed value of μ, let $M(\mu) = \max_{\omega \in [0,\infty]} B$. Determine the behaviour of $M(\mu)$ as $\mu \to 0$. This phenomenon is related to "tuning" in RLC circuits.