MATH 1500, Homework 2

Due: Monday, 28 September. Questions that say "bonus" are optional; however bonus marks will be given if they are correctly attempted.

1. Adams, Section 1.2 questions 2-6 (just an answer suffices), 9, 13,14,18, 22, 32, 33
2. Find the following limits.

$$\lim_{x\to\infty} \frac{x^3+x-5}{3+3x^3}$$
$$\lim_{x\to-\infty} \frac{x^3+x-5\sin x}{3+3x^3}$$
$$\lim_{x\to\infty} \frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{4x+2}}$$
$$\lim_{x\to\infty} \sqrt{x+1}-\sqrt{x}$$

   (a) Consider

$$\lim_{x\to 0^+} \frac{\left(x^{1/2}+2x^2\right)^{1/2}}{x^p}.$$

   Find p such that this limit exists and is non-zero. What is its value?

   (b) Repeat part (a) but with "$x \to 0^+$" replaced with "$x \to +\infty$".

3. (a) Consider the function

$$f(x) = \frac{1}{1-x^2}.$$

   Find the limits

$$\lim_{x\to 1^+} f(x), \quad \lim_{x\to 1^-} f(x), \quad \lim_{x\to\infty} f(x)$$

   Sketch the graph of the function $f(x)$ (note the following useful property: $f(x) = f(-x)$). Indicate vertical and horizontal asymptotes.

   (b) Consider the function

$$f(x) = \frac{1}{(1-x)^2}.$$

   Find the limits

$$\lim_{x\to 1^+} f(x), \quad \lim_{x\to 1^-} f(x), \quad \lim_{x\to\infty} f(x)$$

   Sketch the graph of the function $f(x)$. Indicate vertical and horizontal asymptotes.

4. The purpose of this question is to use the $\delta - \varepsilon$ definition of the limit to prove that

$$\lim_{h\to 0} \frac{1}{2+h} = \frac{1}{2}. \tag{1}$$

   (a) Write down the $\delta - \varepsilon$ definition of the limit, as applied to (1).

   (b) Find a number $\delta > 0$ such that $\left|\frac{1}{2+h} - \frac{1}{2}\right| < 0.1$ whenever $|h| < \delta$.

   (c) Prove (1)

5. [BONUS] Suppose that a function $f(x)$ satisfies $f(x+y) = f(x) + f(y)$ for all x, y.

   (a) Show that $f(0) = 0$.

   (b) Suppose that $\lim_{x\to 0} f(x) = 0$. Use the $\delta - \varepsilon$ definition of limit to show that $\lim_{h\to 0} f(x+h) = f(x)$ for all x.

6. [BONUS] Suppose that $0 \le g(x) \le f(x)$ for all x and suppose that $f \to 0$ as $x \to 0$. Show that $g \to 0$ as $x \to 0$.