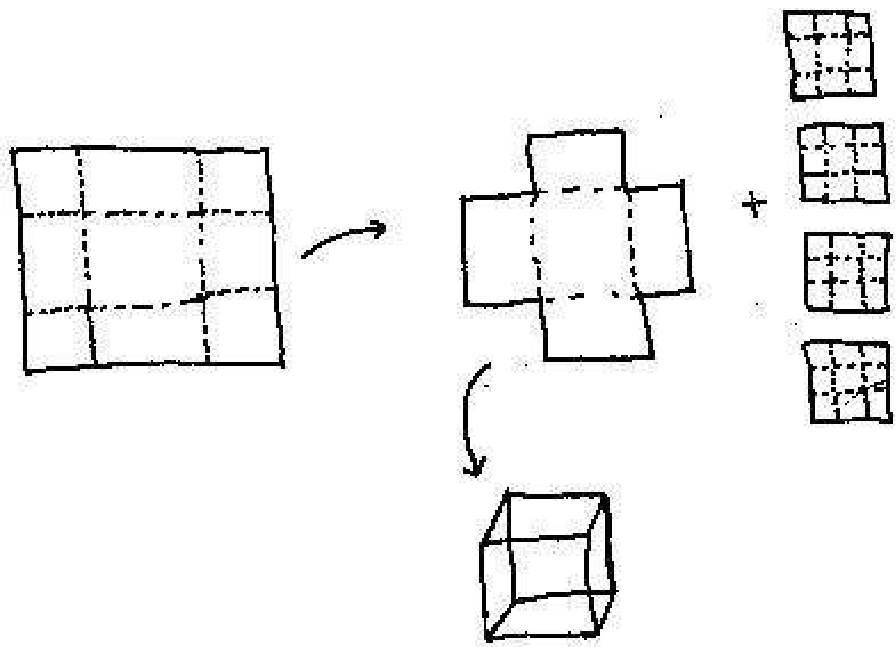
MATH 1500, Homework 6

Due date: 22 October (Friday)

1. [BONUS] Take a square sheet of paper whose side is of length l. From each of the four corners, cut out a smaller square each of whose length is a fraction r_1 of l, with $0 < r_1 < 1/2$. From the resulting cross, make a box of height lr_1 and of base length $l(1-2r_1)$. For each of the remaining four squares, cut out four corners, each of whose length is a fraction r_2 of the length of the small square. Then make four additional boxes from the four resulting crosses.



   (a) How should r_1 and r_2 be chosen in order to maximize the total volume of the resulting five boxes?

   (b) Continue the procedure indefinitely, defining a sequence of ratios $r_1, r_2, r_3 \ldots$ and resulting in $1, 5, 21, \ldots$ boxes. Suppose that it is required that all these ratios are the same: $r = r_1 = r_2 = r_3 = \ldots$. How should you choose r in order to maximize the total volume?

   (c) Now suppose that you are free to choose $r_1, r_2, r_3 \ldots$ independently from one another, in such a way as to maximize the total volume. Would you get a different value for r_1 than what you found for r in part b? If the same, why? If different, what would it be?

2. In class, we showed that $e < 4$. Show that $e < 3$.

3. Show that $\ln(1/x) = -\ln(x)$.

4. In class, we started with $\ln x$ and then defined e^x as its inverse. In this exercise, we will define e^x first, then $\ln x$ from it.

   (a) Assume that there exists a differentiable function $f(x)$ that is defined for all x and that satisfies the equation

$$f'(x) = f(x) \text{ and } y(0) = 1. \tag{1}$$

   Show $f(x) > 0$ for all x. (hint: mean value theorem is useful for this). Conclude that $f(x)$ is increasing.

(b) Show that $f(x+y) = f(x)f(y)$. Hint: you may assume that the solution to (1) is unique [you are asked to show this in (e) as a bonus question].

(c) Show that $f(x) \to \infty$ as $x \to \infty$ and that $f(x) \to 0$ as $x \to -\infty$.

(d) Let $g(x)$ be the inverse of $f(x)$. [such an inverse exists since $f(x)$ is increasing]. Show that $g'(x) = \frac{1}{x}$ and that $g(1) = 0$.
We then call $f(x)$ the "exponential" and $g(x)$ the "logarithm".

(e) [BONUS] Show that the solution to (1) is unique. Hint: if $f_1(x)$ and $f_2(x)$ are two solutions of (1) then consider a new function $u(x) = f_1(x) - f_2(x)$. What equation does $u(x)$ satisfy?

5. Suppose that $x^{g(x)} = g(x)^x$. Find $g'(x)$ in terms of $g(x)$. Given that $g(2) = 4$, find $g'(2)$.

6. A couple wants to morgtage a \$200,000 home with 10% downpayment. They decide to pay it off in 25 years.

   (a) If the interest rate is 5% per year compounded monthly, calculate their monthly payements.

   (b) [BONUS] An ad recently seen on some websites stated: "*\$500,000 mortgage for \$1500 per month!*" If the prime interest rate is 5% per year, show that this is false advertisement.

   (a) You just got a job at GoodPay corp. You decide that you will set aside \$1000 per month into your savings account. The interest rate is 5% per year, compounded monthly. How much will you have in 25 years? How long before you have \$500,000 in your savings account?

   (b) Every year, you get a salary increase of 3%. So you decide to increase the amount you deposit into your savings account by 3% every year. How much will you have in 10 years? How long before you have \$500,000 in your savings account?