Sample practice questions

1. Compute the first three non-zero terms of the Taylor series expansion of the following functions around $x = 0$:

$$f(x) = \sin(x^2)$$
$$f(x) = \sin(x + x^2)$$
$$f(x) = \log(1 + x^2)$$
$$f(x) = \frac{1}{1 + x + x^2}$$

2. Using question 1, find the following limit:

$$\lim_{x \to 0} \frac{\log(1 + x^2) - x^2}{x^4}$$
$$\lim_{x \to 0} \frac{\sin(x + x^2) - x - \log(1 + x^2)}{\sin(x^3)}$$

3. The function $f(x)$ has the following properties: $f(0) = 1, f'(0) = 2, f''(0) = 3$ and $f'''(x) = \frac{1-2x+3x^2-4x^4+5x^5}{5-x^2}$. Estimate $f(1)$ as best as you can, and give the tightest error bound that you can.

4. (a) Expand $f(x) = e^{-x^2}$ around $x = 0$ up to second order.

   (b) Use (a) to estimate $\int_0^{1/2} e^{-x^2}\, dx$.

   (c) Using the Error formula for Taylor series, estimate the maximum error in part (b).

5. Find the following limits:

$$\lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{n} \exp\left(\frac{2j}{n}\right)$$
$$\lim_{n \to \infty} \sum_{j=1}^{n} \frac{n}{j^2 + n^2}$$

6. Evaluate the following integrals:

$$(a) \int x^2 e^x dx \qquad (b) \int_0^1 x\sqrt{1-x}dx \qquad (c) \int_0^{\pi/2} \sin^3 x \; dx$$
$$(d) \int_0^1 \arcsin x \; dx \qquad (e) \int_0^{\pi/3} \tan^4(x) \qquad (f) \int \frac{2x^4 + 5x^3 + 4x^2 + x + 2}{x^3 + 2x^2} dx$$
$$(g) \int \frac{x^2}{(16 - x^2)^{3/2}} dx \qquad (h) \int \frac{1}{(4x^2 - 8x + 8)^2} dx \qquad (i) \int_0^{1/2} \sqrt{x - x^2}$$
$$(j) \int x \ln(x) \; dx \qquad (k) \int \frac{1}{x^3 + x} dx \qquad (l) \int \frac{dx}{4x^2 - 4x + 5}$$