Romberg integration example

Consider

$$\int_1^2 \frac{1}{x} dx = \ln 2.$$

We will use this integral to illustrate how Romberg integration works. First, compute the trapezoid approximations starting with $n = 2$ and doubling n each time:

$n = 1: \; T_1^0 = \left(1 + \frac{1}{2}\right) \frac{1}{2} = 0.75;$
$n = 2: \; T_2^0 = 0.5 \left(\frac{1}{1.5}\right) + \frac{0.5}{2}(1 + \frac{1}{2}) = 0.708333333$
$n = 4: \; T_3^0 = 0.25 \left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}\right) + \frac{0.25}{2}(1 + \frac{1}{2}) = 0.69702380952$
$n = 8: \; T_4^0 = 0.69412185037$
$n = 16: T_5^0 = 0.69314718191.$

Next we use the formula:

$$T_k^i = \frac{4^i T_k^{i-1} - T_{k-1}^{i-1}}{4^i - 1}$$

The easiest way is to keep track of computations is to build a table of the form:

T_1^0				
T_2^0	T_2^1			
T_3^0	T_3^1	T_3^2		
T_4^0	T_4^1	T_4^2	T_4^3	
T_5^0	T_5^1	T_5^2	T_5^3	T_5^3

Starting with the first column (which we just computed), all other entries can be easily computed. For example starting with $T_1^0,\ T_2^0$ we find

$$T_2^1 = \frac{4T_2^0 - T_1^0}{3} = 0.694444$$
$$T_3^1 = \frac{4T_3^0 - T_2^0}{3} = 0.693253; \quad T_3^2 = \frac{16T_3^1 - T_2^1}{15} = 0.69317460$$

and so on. Every entry depends only on its left and left-top neighbour. Continuing in this way, we get the following table:

0.75000000000				
0.70833333333	**0.69**444444444			
0.69702380952	**0.693**25396825	**0.6931**7460317		
0.69412185037	**0.6931**5453065	**0.69314**790148	**0.693147**47764	
0.69339120220	**0.693147**65281	**0.6931471**9429	**0.69314718**307	**0.693147181**91

The correct digits are shown in bold (the exact answer to 15 digits is given by $\ln 2 = 0.693147180559945$). Here is the table listing error $T_i^k - \ln 2$.

5.7e-02				
1.5e-02	1.3e-03			
3.9e-03	1.1e-04	2.7e-05		
9.7e-04	7.4e-06	7.2e-07	3.0e-07	
2.4e-04	4.7e-07	1.4e-08	2.5e-09	1.4e-09

Note that each successive iteration yields around two extra digits (*why*?). The final iteration only required $n = 16$ function evaluations, plus $O(\ln n)$ arithmetic operations to build the table.

Exercise. Use four iterations of Romberg integration to estimate $\pi = \int_0^1 \frac{4}{1+x^2} dx$. Comment on the accuracy of your result.