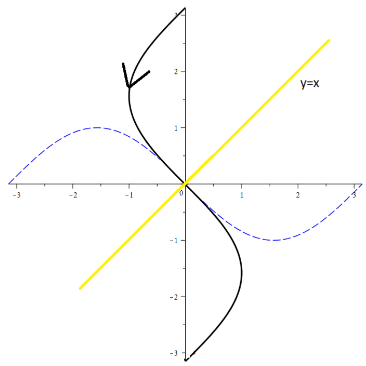
Homework 1

1. Sketch the parametric curve $x = \sin(t)$, $y = -t$, $\quad t = -\pi \ldots \pi$. Use arrows to indicate the direction of increasing t.

   Solution. One way is to eliminate t : $x = \sin(-y)$; $y \in [-\pi, \pi]$. So I'll sketch "$y = \sin(-x)$" and then its inverse is what I want. See below where $y = \sin(-x)$ is indicated by a dashed line, and the actual graph of the curve is indicated by the solid line. The direction is from top to bottom since $y = -t$.

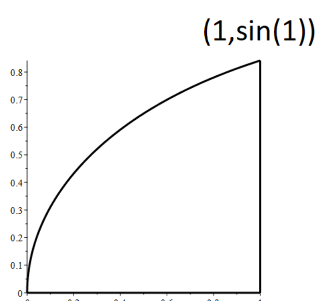


2. Find a parametric equation for a circle of radius 2, centered at $(3, 4)$.

   Solution. $(x, y) = (3, 4) + 2(\cos(t), \sin(t))$, or $\begin{cases} x = 3 + 2\cos t \\ y = 4 + 2\sin t \end{cases}$.

3. (a) Sketch the region enclosed by the curve $x = t^2, y = \sin t$, $0 \le t \le 1$, the x-axis and the line $x = 1$. (b) Find the area of this region.

   Solution. Here is the sketch:

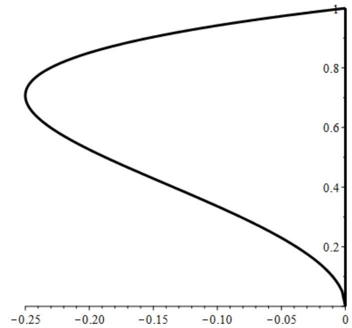


   The area is given by

$$A = \int_{t=0}^{1} y dx = \int_0^1 \sin t(2t) dt = -2t\cos t|_0^1 + \int_0^1 2\cos t$$
$$= -2\cos 1 + 2\sin 1 = 0.60233.$$

4. (a) Sketch the region enclosed by the curve $x = t^2 - t, y = \sqrt{t}$, and the y-axis. (b) Find the area of this region.

Solution. Here is the sketch:



The signed area is given by

$$A = \int_{t=0}^{1} x dy = \int_{0}^{1} (t^2 - t)\left(\frac{1}{2}t^{-1/2}\right) dt = \frac{1}{2}\int_{0}^{1}\left(t^{3/2} - t^{1/2}\right) dt$$
$$= \frac{1}{2}\left(\frac{2}{5} - \frac{2}{3}\right) = -\frac{2}{15}.$$

So the (unsigned) area is $2/15$.

Alternatively, you can use ydx instead:

$$A = \int_{t=0}^{1} ydx = \int_{0}^{1} t^{1/2}(2t-1)\, dt = \frac{2}{15}.$$

(note the signs are opposite if you do ydx or xdy).

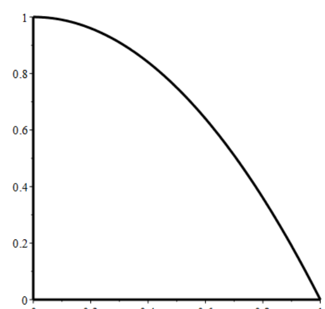
5. Consider a parameteric curve given by $x = \sin t, y = \cos^2 t;\ \ t \in [0, \pi/2]$.

   (a) Sketch this curve.

   (b) Determine the equation of the line tangent to this curve at $t = \pi/2$.

   (c) Determine the area bounded by this curve and the x and y axes.

Solution. Here is the sketch:



(b) We have $(x,y)_{t=\pi/2} = (1,0)$ and $(x',y')_{t=\pi/2} = (\cos(\pi/2), -2\sin\pi/2\cos\pi/2) = (0,0)$. So $t = \pi/2$ is a turning point, and we need to interpret $dy/dx|_{t=\pi/2}$ in terms of a limit:

$$\lim_{t\to\pi/2}\frac{y'(t)}{x'(t)} = \lim_{t\to\pi/2}\frac{-2\sin t\cos t}{\cos t} = \lim_{t\to\pi/2}\frac{-2\sin t}{1} = -2.$$

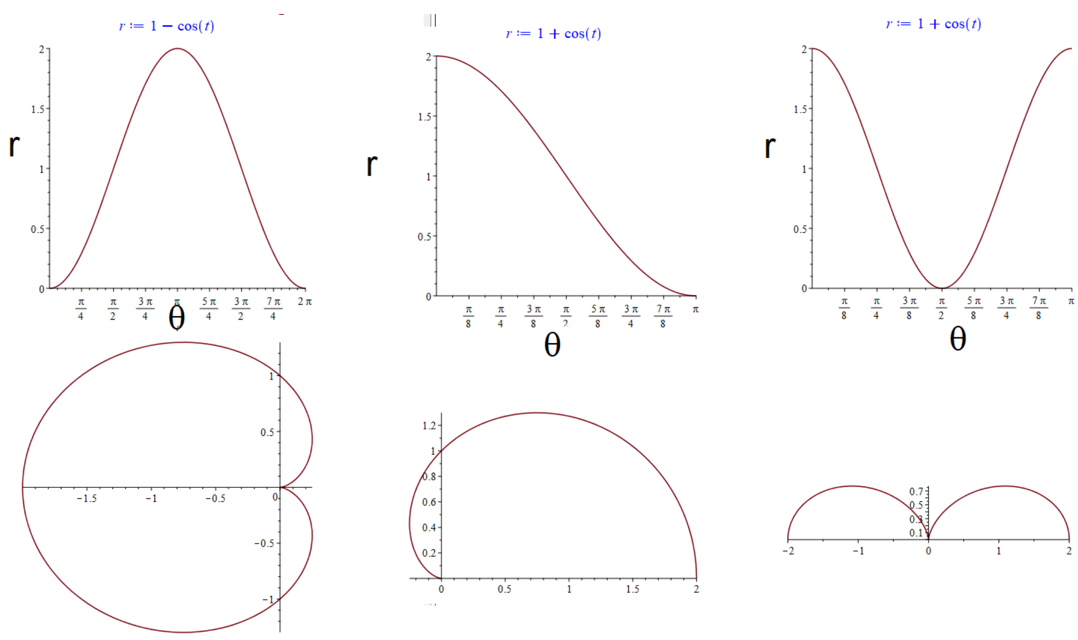
So the slope is $m = 2$ and the line goes through $(1, 0)$; its equation is then $y - 0 = -2(x - 1)$.

(c) To compute the area:

$$A = \int_0^{\pi/2} \sin t (2 \sin t \cos t) dt = (\text{use substitution } u = \sin t)$$
$$= \int_0^1 2u^2 du = \frac{2}{3}.$$

6. Sketch the following curves given in polar coordinates: (a) $r = 1 - \cos\theta, \quad 0 \le \theta \le 2\pi$; (b) $r = 1 + \cos(\theta), \quad 0 \le \theta \le \pi$; (c) $r = 1 + \cos(2\theta), \quad 0 \le \theta \le \pi$.

Solution. Here are the plots.



(a) (b) (c)

7. Find the length of the curve $x = e^{-t} \cos(2t), \; y = e^{-t} \sin(2t), \; t = 0..2$.

Solution. Compute

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$
$$= \sqrt{\left(-e^{-t} \cos(2t) - 2e^{-t} \sin(2t)\right)^2 + \left(-e^{-t} \sin(2t) + 2e^{-t} \cos(2t)\right)^2} dt$$
$$= e^{-t}\sqrt{5} dt$$

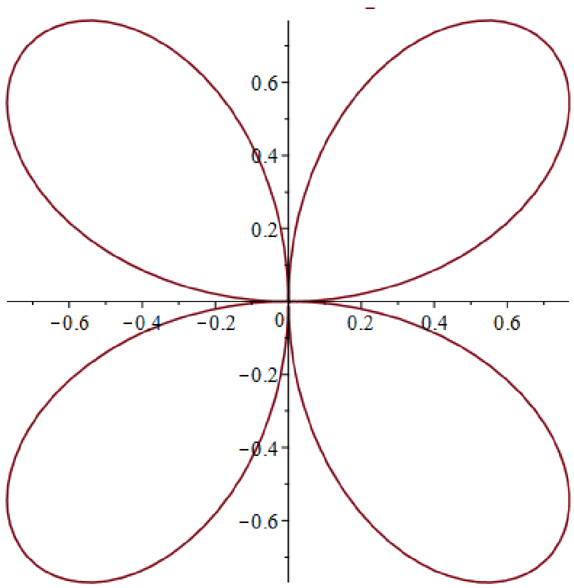
so that the length $L = \int_0^2 e^{-t}\sqrt{5} dt = \sqrt{5}(1 - e^{-2})$.

8. Sketch the curve $\left(x^2 + y^2\right)^3 = 4x^2y^2$ by first converting it to polar coordinates. Hint: $2\cos\theta \sin\theta = \sin(2\theta)$.

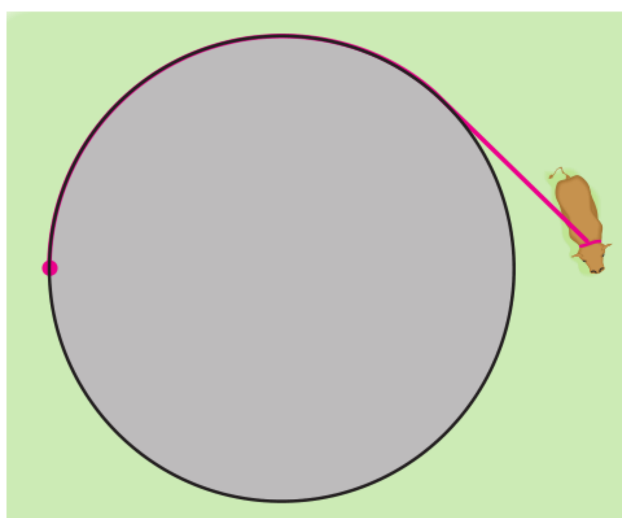
Solution. $x^2 + y^2 = r^2$ and $x^2y^2 = r^4(\cos\theta\sin\theta)^2 = r^4\sin^2(2\theta)/4$ so the curve becomes

$$r^6 = r^4\sin^2(2\theta)$$
$$r = |\sin(2\theta)|$$

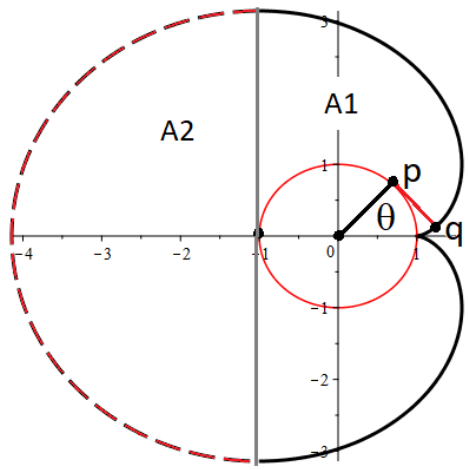
which is sketched here:



9. A cow is tied to a silo with radius 1 by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.



Solution.



We first need to find the boundary of this curve. Let θ be the angle as indicated. Then the rope length pq is also θ (since that's exactly the arclength of the segment between point p and the x-axis), and the line through points p and q is tangent to the circle at p. The point p is on the unit circle so we parameterize it as $p = (\cos\theta, \sin\theta)$. The vector pointing from p to q is in the direction of tangent vector at $p : p' = (-\sin\theta, \cos\theta)$. The unit vector pointing from p to q is given by $v = \pm p'/|p|$. so that here, $v = (\sin\theta, -\cos\theta)$. Therefore $q = (x, y) = p + \theta v$:

$$\begin{array}{l} x = \cos\theta + \theta\sin\theta \\ y = \sin\theta - \theta\cos\theta \end{array}, \quad \theta \in (0, \pi).$$

When $\theta = \pi$, the rope is fully stretched. To the left of this, the rope pivots around $(-1, 0)$ and goes along a circle of radius π. So the total area is:

$$A = 2(A_1 + A_2)$$

where $A_1 = \left|\int_0^\pi y dx\right| - \pi/2$ and where $A_2 = \frac{\pi^3}{4}$ (the area of quarter-circle or radius π). Evaluate A_1 :

$$A_1 = \left|\int_0^\pi (\sin\theta - \theta\cos\theta)\,\theta\cos\theta d\theta\right| - \pi/2 = \left|-\frac{\pi^3}{6} - \frac{\pi}{2}\right| - \frac{\pi}{2} = \frac{\pi^3}{6}$$

so the total area is

$$A = 2\left(\frac{\pi^3}{6} + \frac{\pi^3}{4}\right) = \frac{5\pi^3}{3} = 51.677.$$

Remark: if you try $\left|\int_0^\pi x dy\right|$ instead of $\left|\int_0^\pi y dx\right|$, you'd get a different (and wrong) asnwer. The reason for this, is that strictly speaking, to find the area of A_1, you need to integrate along a closed curve. The two other boundaries are vertical line $x = -1, y = t, t = \pi \ldots 0$ and the horizontal line $y = 0$. Along the horizontal line, both $\int y dx$ and $\int x dy$ give zero. But along the vertical, $\int x dy$ gives $\int_{t=\pi}^0 (-1) dt = \pi$ whereas $\int y dx$ gives zero. This is why the choice $\left|\int_0^\pi y dx\right|$ is more convenient (it automatically gives zero along both of the other boundaries).