Homework 2

1. (a) Sketch the the curves $r = \sin\theta$ and $r = \sqrt{3}\cos\theta$.

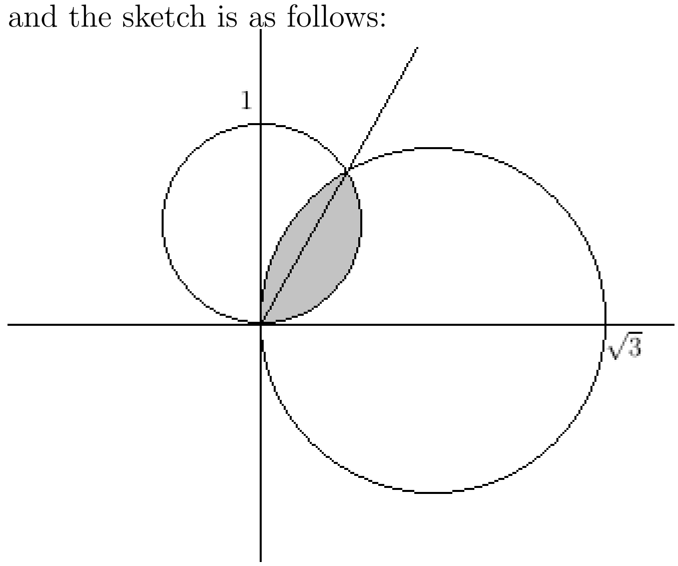
   (b) Find the area of the region that lies inside the two curves.

   Solution:

   (a) The two curves intersect when

$$\begin{aligned} \sin\theta &= \sqrt{3}\cos\theta \\ \tan\theta &= \sqrt{3} \\ \theta &= \pi/3 \end{aligned}$$

   and the sketch is as follows:



   (b) The area is given by

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}\sin^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sqrt{3}\cos\theta)^2 \, d\theta \\ &= \int_0^{\pi/3} \frac{1}{4}(1-\cos(2\theta)) \, d\theta + \int_{\pi/3}^{\pi/2} \frac{3}{4}(1+\cos(2\theta)) \, d\theta \\ &= \frac{1}{4}(\theta - \frac{\sin(2\theta)}{2})\Big|_0^{\pi/3} + \frac{3}{4}(\theta + \frac{\sin(2\theta)}{2})\Big|_{\pi/3}^{\pi/2} \\ &= \frac{1}{4}(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) + \frac{3}{4}(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}) = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}. \end{aligned}$$

2. (a) Find an equation of the sphere that passes through the point $(-2, 3, 0)$ and has center $(2, -1, 2)$.

   (b) Describe the curve in which this sphere intersects the plane $y = 1$.

   Solution:

   (a) The radius of the sphere will be equal to the distane of the two points, i.e. equal to $\sqrt{(-2-2)^2 + (3+1)^2 + (-2)^2} = \sqrt{36} = 6$. So an equation for the sphere is

   $$(x-2)^2 + (y+1)^2 + (z-2)^2 = 36$$

   (b) We set $y = 1$ in the above equation and we get

   $$(x-2)^2 + (1+1)^2 + (z-2)^2 = 36 \text{ or } (x-2)^2 + (z-2)^2 = 32.$$

   So the curve is given by

   $$\begin{cases} (x-2)^2 + (z-2)^2 = 32 \\ y = 1 \end{cases}$$

   which is a circle on the plane $y = 1$ with center $(2, 1, 2)$ and radius $\sqrt{32}$.

3. Find two unit vectors that are orthogonal to both $\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.
   Solution: Their cross product

   $$(\mathbf{j}+2\mathbf{k}) \times (\mathbf{i}-2\mathbf{j}+3\mathbf{k}) = (3-(-4))\mathbf{i} - (0-2)\mathbf{j} + (0-1)\mathbf{k} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

   is a vector that is orthogonal to both given vectors. So now we just need to find the unit vector on the same direction with it. It's length is

   $$\sqrt{7^2 + 2^2 + (-1)^2} = \sqrt{54} = 3\sqrt{6},$$

   so the unit vector in the same direction is $\frac{1}{3\sqrt{6}}(7\mathbf{i} + 2\mathbf{j} - \mathbf{k})$. Also its opposite $-\frac{1}{3\sqrt{6}}(7\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ is a unit vector orthogonal to both given vectors.

4. (a) Find the unit vectors that are parallel to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6, 1)$.

(b) Find the unit vectors that are perpendicular to the tangent line.

(c) Sketch the curve $y = 2\sin x$ and the vectors in parts (a) and (b), all starting at $(\pi/6, 1)$.

Solution:

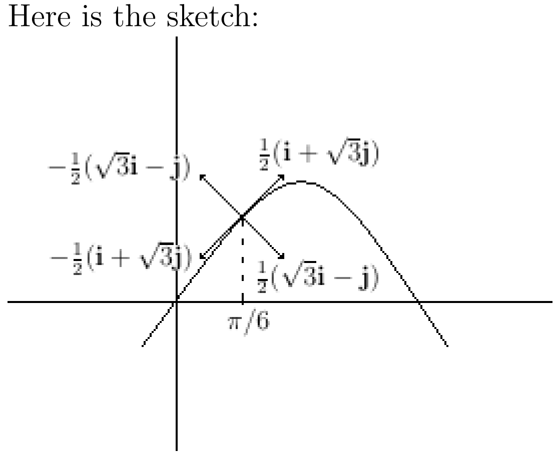
(a) The slope of the tangent line to the given curve at $(\pi/6, 1)$ is

$$\left.\frac{dy}{dx}\right|_{\pi/6} = 2\cos x\Big|_{\pi/6} = \sqrt{3},$$

so a parallel vector is $\mathbf{i} + \sqrt{3}\mathbf{j}$, with length $\sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$. So the unit vectors parallel to the tangent line are $\frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$ and $-\frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$.

(b) Since the slope of the tangent line is $\sqrt{3}$, the slope of the line perpendicular to it has slope $-\frac{1}{\sqrt{3}}$. So a vector perpendicular to the tangent line is $\sqrt{3}\mathbf{i} - \mathbf{j}$, with length $\sqrt{\sqrt{3}^2 + (-1)^2} = 2$. So the unit vectors perpendicular to the tangent line are $\frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$ and $-\frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$.

(c) Here is the sketch:



5. (a) Find two unit vectors that make an angle of 60° with $\mathbf{v} = \langle 3, 4\rangle$.

(b) Find the acute angle between the curves $y = \sin x$ and $y = \cos x$ at their point of intersection for $0 \le x \le \pi/2$. (The angle between two curves is the angle between their tangent lines at the point of interection.)

Solution:

(a) Let $u = \langle u_1, u_2 \rangle$ be a unit vector that makes an angle of 60° with $\mathbf{v} = \langle 3, 4 \rangle$. Then $u \cdot v = |u||v| \cos 60^\circ = 1 \cdot 5 \cdot 1/2 = 5/2$. However, $u \cdot v = 3u_1 + 4u_2$. So

$$3u_1 + 4u_2 = 5/2.$$

Also, since u is a unit vector,

$$u_1^2 + u_2^2 = 1.$$

Solving the system of the two equations we get the solutions

$$\begin{cases} u_1 = \frac{3+4\sqrt{3}}{10} \\ u_2 = \frac{4-3\sqrt{3}}{10} \end{cases} \quad \text{or} \quad \begin{cases} u_1 = \frac{3-4\sqrt{3}}{10} \\ u_2 = \frac{4+3\sqrt{3}}{10} \end{cases}$$

So the two unit vectors are $\langle \frac{3+4\sqrt{3}}{10}, \frac{4-3\sqrt{3}}{10} \rangle$ and $\langle \frac{3-4\sqrt{3}}{10}, \frac{4+3\sqrt{3}}{10} \rangle$.

(b) The intersection point of the two curves is at

$$\begin{aligned} \sin x &= \cos x \\ \tan x &= 1 \\ x &= \frac{\pi}{4}. \end{aligned}$$

So the tangent line to the curce $y = \sin x$ at $\pi/4$ has slope $\frac{d \sin x}{dx}\Big|_{\pi/4} = \cos \pi/4 = \frac{\sqrt{2}}{2}$ and the tangent line to the curve $y = \cos x$ at $\pi/4$ has slope $\frac{d \cos x}{dx}\Big|_{\pi/4} = -\sin \pi/4 = -\frac{\sqrt{2}}{2}$. So vectors parallel to these tangent lines are $\langle 1, \frac{\sqrt{2}}{2} \rangle$ and $\langle 1, -\frac{\sqrt{2}}{2} \rangle$ respectively. If θ is the acute angle between them, then

$$\cos \theta = \frac{\langle 1, \frac{\sqrt{2}}{2} \rangle \cdot \langle 1, -\frac{\sqrt{2}}{2} \rangle}{|\langle 1, \frac{\sqrt{2}}{2} \rangle||\langle 1, -\frac{\sqrt{2}}{2} \rangle|} = \frac{1 - 1/2}{\sqrt{3/2}\sqrt{3/2}} = \frac{1}{3}.$$

So $\theta = \cos^{-1} \frac{1}{3} \approx 70.5^\circ$.

6. Show that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then the vectors $\mathbf{u}$ and $\mathbf{v}$ must have the same length.
Solution: If $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal then $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$. However
$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}-|\mathbf{v}|^2\\ &= |\mathbf{u}|^2-|\mathbf{v}|^2.\end{aligned}$$
So $|\mathbf{u}|^2-|\mathbf{v}|^2=0$, i.e. $|\mathbf{u}|=|\mathbf{v}|$, since length is positive.

7. Use the scalar triple product to determine whether the points $A = (1,3,2)$, $B = (3,-1,6)$, $C = (5,2,0)$, and $D = (3,6,-4)$ lie in the same plane.
Solution: We need to check whether the three vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ lie in the same plane, i.e. we need to check whether
$$\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = 0.$$
We have $\vec{AB} = \langle 2,-4,4\rangle$, $\vec{AC} = \langle 4,-1,-2\rangle$ and $\vec{AD} = \langle 2,3,-6\rangle$. So
$$\begin{aligned}\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = \begin{vmatrix}2 & -4 & 4\\ 4 & -1 & -2\\ 2 & 3 & -6\end{vmatrix} &= 2\begin{vmatrix}-1 & -2\\ 3 & -6\end{vmatrix} - (-4)\begin{vmatrix}4 & -2\\ 2 & -6\end{vmatrix} + 4\begin{vmatrix}4 & -1\\ 2 & 3\end{vmatrix}\\ &= 24-80-56=0\end{aligned}$$
So the points A, B, C and D all lie in the same plane.

8. Given the points $A = (1,0,1)$, $B = (2,3,0)$, $C = (-1,1,4)$, and $D = (0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.
Solution: We have that $\vec{AB} = \langle 1,3,-1\rangle$, $\vec{AC} = \langle -2,1,3\rangle$ and $\vec{AD} = \langle -1,3,1\rangle$. So the volume of the parallelepiped is given by the absolute value of
$$\begin{aligned}\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = \begin{vmatrix}1 & 3 & -1\\ -2 & 1 & 3\\ -1 & 3 & 1\end{vmatrix} &= \begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix} - 3\begin{vmatrix}-2 & 3\\ -1 & 1\end{vmatrix} - \begin{vmatrix}-2 & 1\\ -1 & 3\end{vmatrix}\\ & -8-3+5=-6.\end{aligned}$$
So the volume is 6.