Homework 3

1. Consider a triangle whose vertices are $P = (1,0,0)$, $Q = (0,1,0)$ and $R = (0,0,2)$.

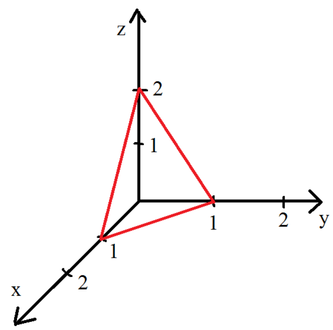
   (a) Sketch the triangle PQR in three dimensional space.

   (b) Find a vector that is in the direction perpendicular to the plane through PQR.

   (c) Find the area of triangle PQR.

   (d) Find the equation of the line that is perpendicular to the plane through PQR and goes through the point $(1,1,1)$.

   Solution. (a)



   (b) Let $u = Q - P = (-1,1,0)$, $v = R - P = (-1,0,2)$. Then

$$u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2\hat{\imath} + 2\hat{\jmath} + \hat{\mathbf{k}}$$

   which is the direction perpendicular to the plane.

   (c) $|u \times v|/2 = \sqrt{9}/2 = 3/2$.

   (d) It's the line in the direction of $u \times v$ going through the point $(1,1,1)$:

$$(x,y,z) = (1,1,1) + (2,2,1)\,t.$$

2. Let P be a point not on the plane that passes through the points Q, R, S.

   (a) Show that the distance d from to the point P to the plane through Q, R, S is

$$d = \frac{|a \cdot (b \times c)|}{|a \times b|}$$

   where $a = QR,\ b = QS$, and $c = QP$.

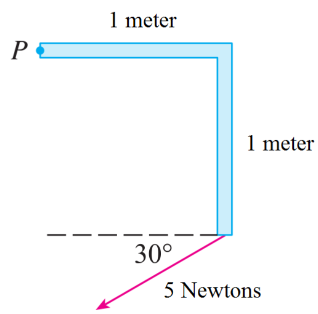
   Solution. The volume $|a \cdot (b \times c)|$ of the paralellopiped spanned by vectors a, b, c equals to the distance d from the point P to the plane QRS, multiplied by the area $|a \times b|$ of the base: $|a \cdot (b \times c)| = d\,|a \times b|$, which is precisely the above formula

   (b) Apply this formula in the case where $P = (0,0,0)$, $Q = (1,0,0)$, $R = (0,1,0)$, and $S = (0,0,2)$.

Here, $a = (-1, 1, 0)$, $b = (-1, 0, 2)$ and $c = (1, 0, 0)$. Then

$$\begin{aligned} b \times c &= 2\hat{\jmath}, \\ |a \cdot (b \times c)| &= 2, \\ a \times b &= -\hat{\mathbf{k}} + 2\hat{\jmath} + \hat{\imath} \\ |a \times b| &= \sqrt{6} \\ d &= 2/\sqrt{6}. \end{aligned}$$

3. Find the magnitude of the torque about P if a 5-N force is applied as shown. Also find the direction of the torque vector.



Solution. The position vector is $r = (1, -1, 0)$ and has magnitude of $|r| = \sqrt{2}$. The force vector has magnitude of $|F| = 5$. The angle in between them is $\theta = 45 + 30 = 75$ degrees. So the magnitude of the torque is $|\tau| = 5 * \sqrt{2} * \sin 75^0 = 6.83$. The right hand rule shows that $r \times F$ is oriented *into* the page (i.e. negative $\hat{\mathbf{k}}$ direction). So $\tau = -6.83\hat{\mathbf{k}}$.

4. The points $P = (0, 0, 0)$, $Q = (1, -1, 1)$, $R = (1, 0, 0)$ and $S = (1, 2, a)$ all lie on the same plane. Determine the value of a. Hint: use the formula for the volume of a parallelepiped.

Solution. Solution 1: Let $u = Q - P = (1, -1, 1)$, $v = R - P = (1, 0, 0)$, $w = S - P = (1, 2, a)$ Then

$$\begin{aligned} u \times v &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (0, 1, 1) \\ w \cdot (u \times v) &= 2 + a \end{aligned}$$

The three vectors are coplanar if and only if $w \cdot (u \times v) = 0$ so that $a = -2$.

5. (a) Find the equation of the line which is parallel to the planes $x + y + z = 1$ and $x = 2$, and which goes through the origin.

(b) Find the equation of the line which is the intersection of the planes in part (a).

Solution. (a) The normals to the two planes are $(1, 1, 1)$ and $(1, 0, 0)$. Their cross product is

$$(1, 1, 1) \times (1, 0, 0) = (\hat{\imath} + \hat{\mathbf{k}} + \hat{\jmath}) \times \hat{\imath} = \hat{\jmath} - \hat{\mathbf{k}}$$

So the desired line is in the direction $(0,1,-1)$ going through the origin:

$$(x,y,z) = (0,0,0) + t\,(0,1,-1)\,.$$

(b) We just need any point which is in the intersection of the two planes, i.e. any solution to $x+y+z=1$ and $x=2$. For example $x=2, y=0, z=-1$. So the desired line goes through $(2,0,-1)$ and has the same direction as found in part (a):

$$(x,y,z) = (2,0,-1) + t\,(0,1,-1)\,.$$

6. Determine whether the lines

$$L_1 : x = \frac{y-1}{-1} = \frac{z-2}{3}, \qquad L_2 : \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}$$

are parallel, intersecting, or skew. If they intersect, find the point of intersection.

Solution. The direction vector of the first line is $(1,-1,3)$ and of the second line is $(2,-2,7)$ which are not scalar multiples of each other, so the lines are not parallel. Do they intersect? If they did, all four equations above would need to be satisfied simultaneously; that is:

$$\begin{aligned}
-x-y+1 &= 0\\
3x-z+2 &= 0\\
x+y-5 &= 0\\
x-2-\frac{2}{7}z &= 0
\end{aligned}$$

In fact, this system has no solution: the simplest way to see that is that equation 1 and 3 above are inconsistent. So this sysem is skew.

7. (a) Find equations for the line of intersection of the planes $3x-2y+z-1=0,\ 2x+y-3z-3=0$. (b) Find the acute angle between these planes.

8. For each of the conic below, indentify its type and sketch it. If it's an ellipse/circle make sure to indicate its radii and center. For hyperbolas, indicate their center (and directions).

(a) $x^2+4x+y^2=0$

(b) $4x^2-8x+y^2-4y=1$

(c) $4x^2-8x-y^2+4y=4$

(d) $-4x^2+8x+y^2-4y=4$

Solution. Completing the squares, we rewrite these quadratics as follows:

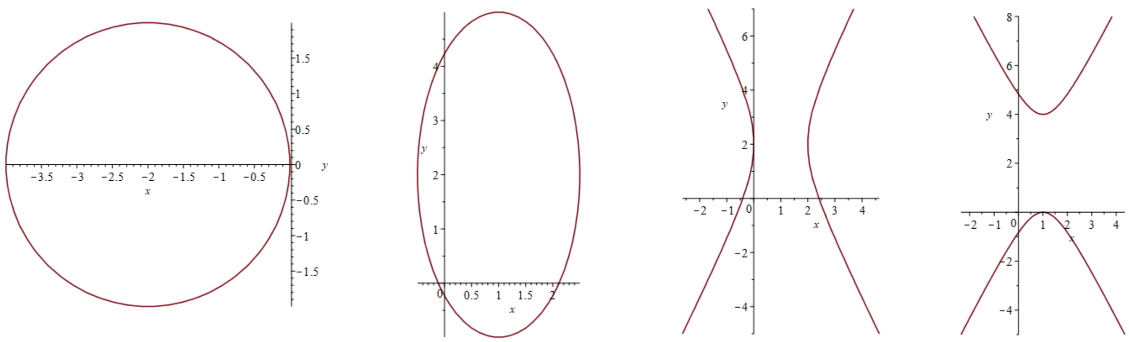
(a) $(x+2)^2+y^2=4$,

(b) $4\,(x-1)^2+(y-2)^2=9$, :

(c) $4\,(x-1)^2-(y-2)^2=4$ and

(d) $-4\,(x-1)^2+(y-2)^2=4$.

So (a) is a circle of radius 2 centered at (-2,0). For (b), we write it as $\frac{(x-1)^2}{(3/2)^2} + \frac{(y-2)^2}{3^2} = 1$, so it's an ellipse centered at (1,2) with horizontal radius of 3/2 and vertical radius of 3. For (c), it's a hyperbola centered at (1,2), opening sideways, going through $y = 2$ at $x = 0, 2$. For (d), it's a hyperbola centered at (1,2) opening vertically, going through $x = 1$ when $y = 0, 4$.



(a) (b) (c) (d)

9. Sketch the graphs of the following quadratic surfaces. For each of these, classify them (e.g. one-sheet hyperboloid, two-sheet hyperboloid, paraboloid, ellisposoid, saddle...).

(a) $x^2 - y^2 + z^2 = 1$

(b) $x^2 - y^2 + z^2 = -1$

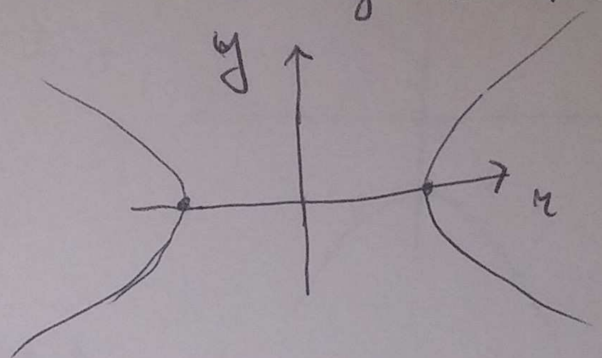
(c) $x^2 - y + z^2 = 1$

(d) $x^2 - 2x - \frac{y^2}{4} + z^2 = 0$

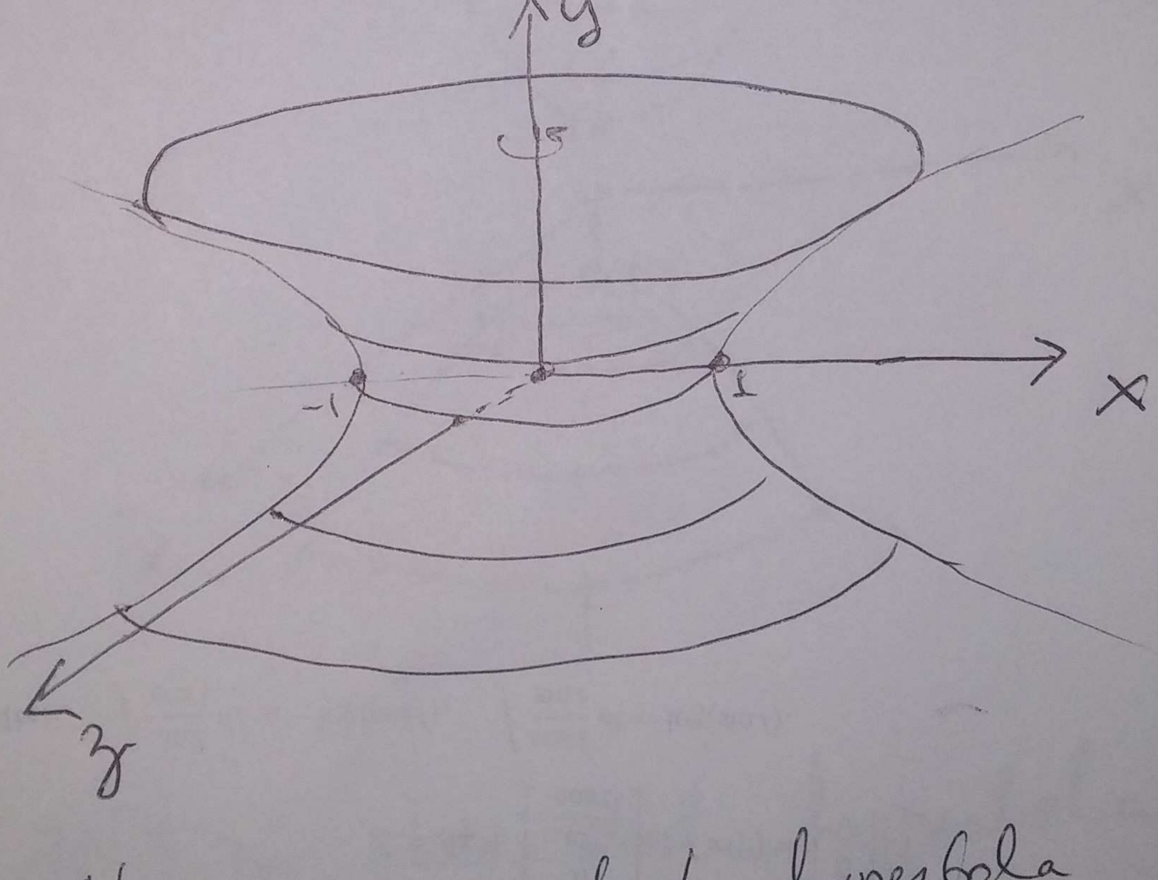
Solutions:

(a) $\underbrace{x^2+z^2}_{r^2} - y^2 = 1$

Let $r = \sqrt{x^2+z^2}$, then $r^2 - y^2 = 1$ has the graph:



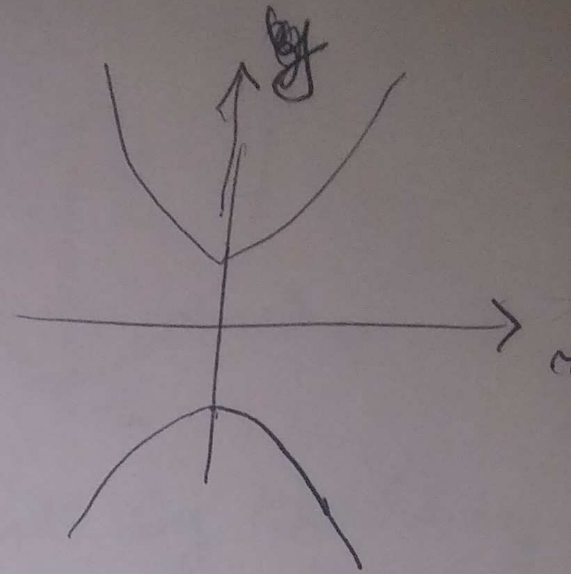
Here, "r" can be thought as x-z plane and the desired surface is the above ~~of~~ hyperbola ~~revol~~ revolved around the y-axis:



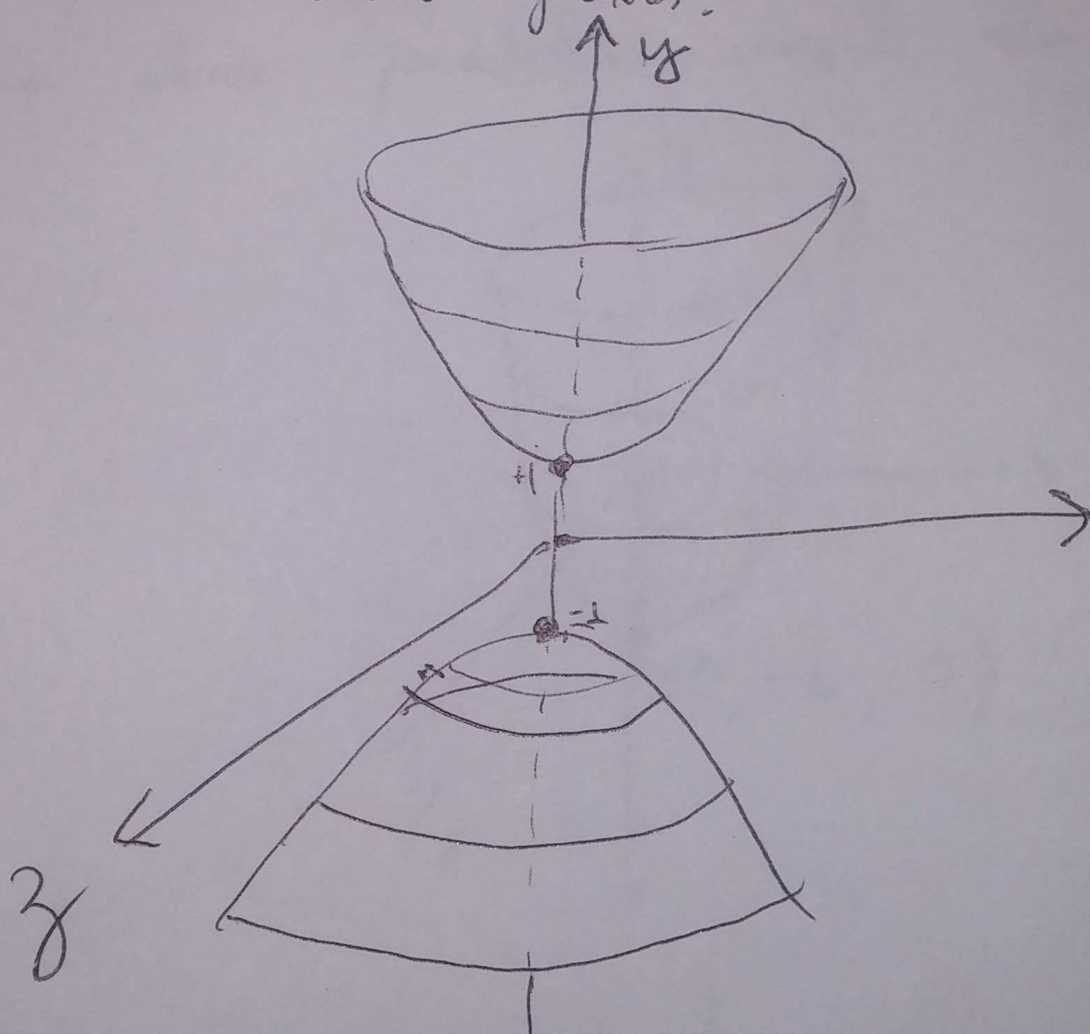
It's a one-sheet hyperbola

(b) $\underbrace{x^2+z^2}_{r^2} - y^2 = -1$

Sketch $r^2 - y^2 = -1$:

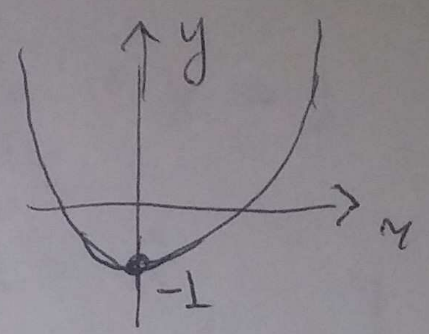


Then revolve about y-axis:

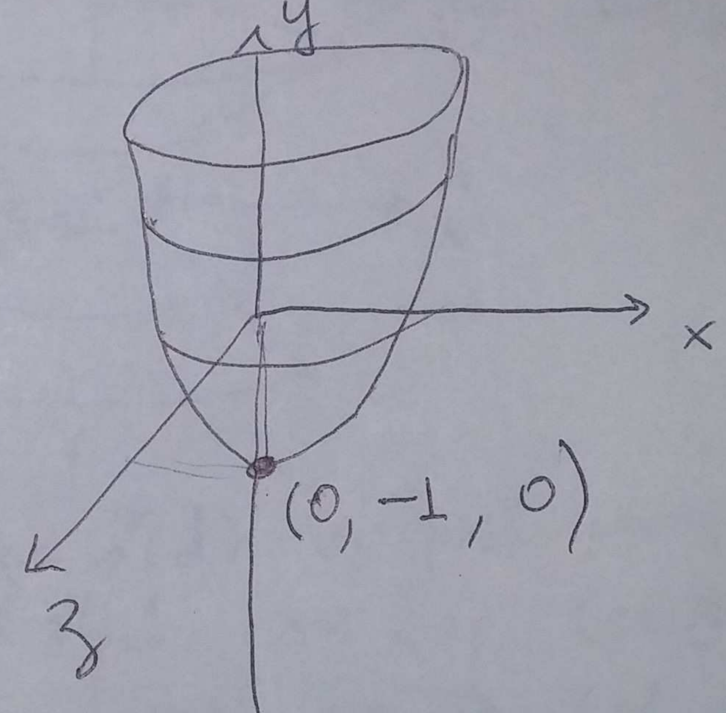


It's a two-sheet hyperbola.

(c) $y = \underbrace{x^2 + z^2}_{r^2} - 1 \rightarrow y = r^2 - 1$:



So the surface is obtained by revolving the above parabola about the y-axis:



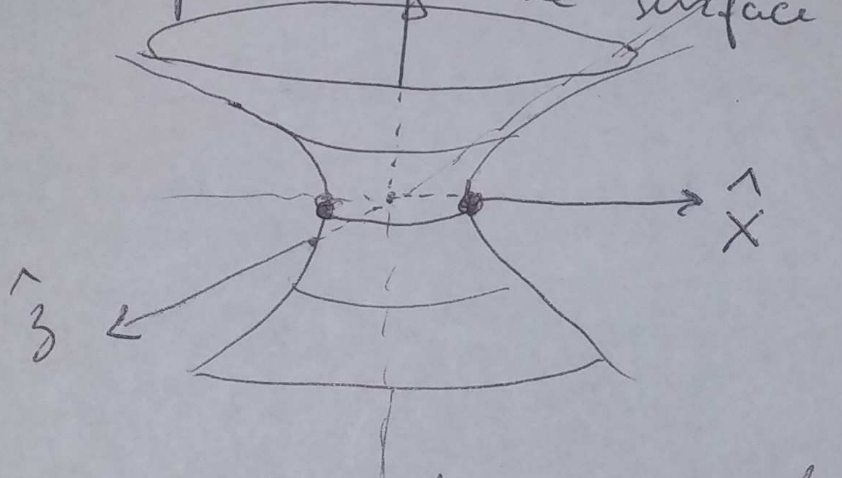
It's a <u>paraboloid</u>

(d) <u>Complete square:</u> $x^2 - 2x = (x-1)^2 - 1$

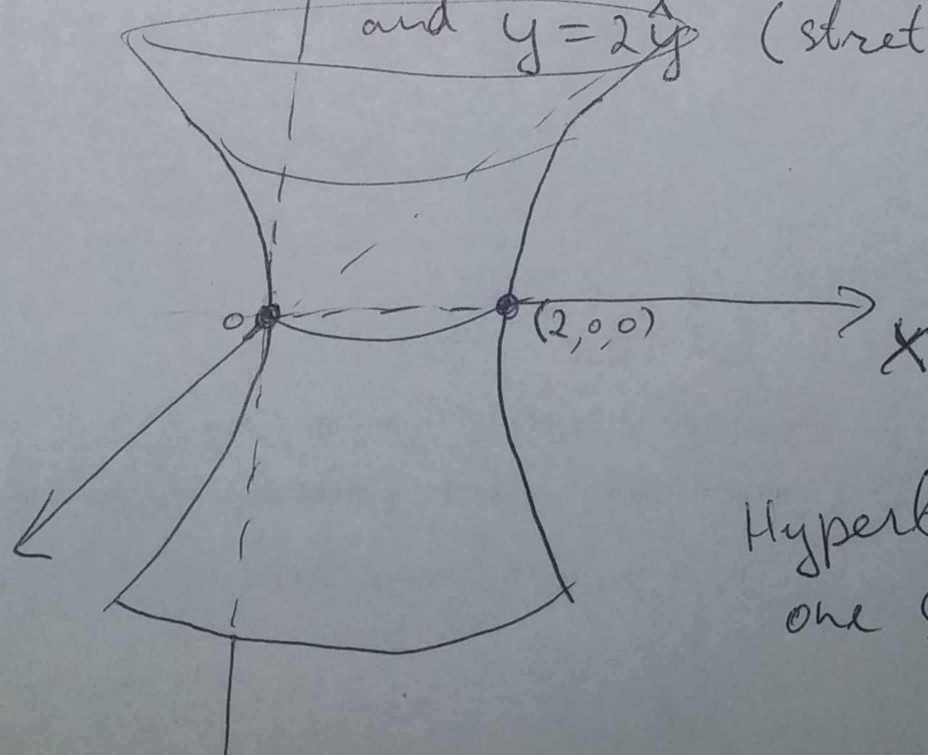
so we get $(x-1)^2 - \frac{y^2}{4} + z^2 = 1$

<u>Change var:</u> $x - 1 = \hat{x}$, $\frac{y}{2} = \hat{y}$, $z = \hat{z}$

then we get $\hat{x}^2 - \hat{y}^2 + \hat{z} = 1$ which is precisely the surface sketched in part (a):



So the surface (d) is just above figure, ~~excha~~ except that $x = \hat{x} + 1$ (shift right) and $y = 2\hat{y}$ (stretch along y-dir):



Hyperboloid of one sheet.