Homework 4

1. (a) Find a vector function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the cylinder $y = x^2$.

   (b) Find the derivative of the curve you found in (a) and the unit tangent vector at $t = 1$.

   Solution

   (a) In order to find the curve of intersection we plug $y = x^2$ in $z = 4x^2 + y^2$ and we get $z = 4x^2 + (x^2)^2 = 4x^2 + x^4$. Then we choose the parameter $x = t$ and we have $y = t^2$ and $z = 4t^2 + t^4$. So the vector function will be $r(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2 + t^4)\,\mathbf{k}$.

   (b) The derivative of $r(t)$ is

   $$r'(t) = \mathbf{i} + 2t\,\mathbf{j} + (8t + 4t^3)\,\mathbf{k}.$$

   The tangent vector at $t = 1$ is $r'(1) = \mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$ and the unit tangent vector is

   $$T = \frac{r'(1)}{|r'(1)|} = \frac{\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}}{\sqrt{1 + 4 + 144}} = \frac{\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}}{\sqrt{149}}.$$

2. Find the point on the curve $r(t) = \langle 2\cos t, 2\sin t, e^t \rangle$, $0 \le t \le \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

   Solution

   We have $r'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$. The tangent vector to the curve is parallel to the plane if and only if it is orthogonal to the plane's normal vector. The plane's normal vector is $\langle \sqrt{3}, 1, 0 \rangle$. So we require

   $$\begin{aligned} \langle -2\sin t, 2\cos t, e^t \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle &= 0 \\ -2\sqrt{3}\sin t + 2\cos t &= 0 \\ \sqrt{3}\sin t &= \cos t \\ \tan t &= \frac{1}{\sqrt{3}} \end{aligned}$$

   and since $0 \le t \le \pi$, we get $t = \frac{\pi}{6}$. So the point on the curve where the tangent is parallel to the plane is $(2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}, e^{\frac{\pi}{6}}) = (\sqrt{3}, 1, e^{\frac{\pi}{6}})$.

3. Evaluate the integrals:

   (a) $\int (t\,\mathbf{i} + \frac{1}{t+1}\,\mathbf{j} - t\ln t\,\mathbf{k})\,dt$

   (b) $\int_0^{\pi/4} (\sec^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + 2t\,\mathbf{k})\,dt$

   Solution

   (a)

$$
\begin{aligned}
&(\int t\,dt)\,\mathbf{i} + (\int \frac{1}{t+1}\,dt)\,\mathbf{j} - (\int t\ln t\,dt)\,\mathbf{k}\\
&= \frac{t^2}{2}\,\mathbf{i} + \ln(t+1)\,\mathbf{j} - (\frac{t^2}{2}\ln t - \int \frac{t^2}{2t}\,dt)\,\mathbf{k}\\
&= \frac{t^2}{2}\,\mathbf{i} + \ln(t+1)\,\mathbf{j} - (\frac{t^2}{2}\ln t - \int \frac{t}{2}\,dt)\,\mathbf{k}\\
&= \frac{t^2}{2}\,\mathbf{i} + \ln(t+1)\,\mathbf{j} - (\frac{t^2}{2}\ln t - \frac{t^2}{4})\,\mathbf{k} + \mathbf{C},
\end{aligned}
$$

   where the third integral has been evaluated with integration by parts and $\mathbf{C}$ is a vector constant.

   (b)

$$
\begin{aligned}
&\int_0^{\pi/4} (\sec^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + 2t\,\mathbf{k})\,dt\\
&= (\int_0^{\pi/4} \sec^2 t\,dt)\,\mathbf{i} + (\int_0^{\pi/4} \cos^2 t\,dt)\,\mathbf{j} + (\int_0^{\pi/4} 2t\,dt)\,\mathbf{k})\\
&= \tan t\Big|_0^{\pi/4}\,\mathbf{i} + \big(\int_0^{\pi/4} \frac{1}{2}(1+\cos(2t))\,dt\big)\,\mathbf{j} + t^2\Big|_0^{\pi/4}\,\mathbf{k}\\
&= (\tan\frac{\pi}{4} - \tan 0)\,\mathbf{i} + \big(\frac{1}{2}(t + \frac{\sin(2t)}{2})\Big|_0^{\pi/4}\big)\,\mathbf{j} + \frac{\pi^2}{16}\,\mathbf{k}\\
&= \mathbf{i} + \frac{1}{2}(\frac{\pi}{4} + \frac{\sin\frac{\pi}{2}}{2} - \frac{\sin 0}{2})\,\mathbf{j} + \frac{\pi^2}{16}\,\mathbf{k}\\
&= \mathbf{i} + \frac{1}{2}(\frac{\pi}{4} + \frac{1}{2})\,\mathbf{j} + \frac{\pi^2}{16}\,\mathbf{k}\\
&= \mathbf{i} + (\frac{\pi}{8} + \frac{1}{4})\,\mathbf{j} + \frac{\pi^2}{16}\,\mathbf{k}
\end{aligned}
$$

4. (a) Find the unit tangent and the unit normal vectors $T(t)$ and $N(t)$ of the curve $r(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$ for $t > 0$.

(b) Find the curvature of $r(t)$.

(c) Find the curvature at the point $(\frac{\pi^2}{4}, 1, \frac{\pi}{2})$.

Solution

(a) We have

$$r'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle,$$

so

$$|r'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}\, t, \text{ since } t > 0.$$

Then

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{5}\, t}\langle 2t, t\sin t, \cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle.$$

and

$$T'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle.$$

So

$$|T'(t)| = \frac{1}{\sqrt{5}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}\sqrt{1} = \frac{1}{\sqrt{5}}$$

and lastly

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{1}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle.$$

(b) The curvature is

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}\, t} = \frac{1}{5t}.$$

(c) The point $(\frac{\pi^2}{4}, 1, \frac{\pi}{2})$ corresponds to $t = \frac{\pi}{2}$ and we have

$$\kappa(\frac{\pi}{2}) = \frac{1}{5\frac{\pi}{2}} = \frac{2}{5\pi}.$$

5. Find the curvature of $y = x^4$ at the point $(\frac{1}{2}, \frac{1}{16})$.
Solution
We have $y' = 4x^3$, $y'' = 12x^2$ and by the formula 13.3.11,

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|12x^2|}{(1 + 16x^6)^{3/2}}.$$

So the curvature at the point $(\frac{1}{2}, \frac{1}{16})$ is

$$\kappa(\frac{1}{2}) = \frac{|12\frac{1}{4}|}{(1 + 16\frac{1}{64})^{3/2}} = \frac{3}{(1 + \frac{1}{4})^{3/2}} = \frac{3}{(\frac{5}{4})^{3/2}} = \frac{24}{\sqrt{5}^3}.$$

6. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.
Solution
If a particle moves with constant speed then $|v(t)| = c$, for some constant c. The acceleration is given by $a(t) = v'(t)$. In order to show that the velocity and the acceleration are orthogonal we need to show that $v(t) \cdot v'(t) = 0$. However,

$$\frac{d}{dt}\big(v(t) \cdot v(t)\big) = v'(t) \cdot v(t) + v(t) \cdot v'(t) = 2v'(t) \cdot v(t)$$

and on the other hand,

$$v(t) \cdot v(t) = |v(t)|^2 = c^2.$$

So

$$\frac{d}{dt}\big(v(t) \cdot v(t)\big) = \frac{d}{dt}c^2 = 0$$

and then $2v'(t) \cdot v(t) = 0$, i.e. $v'(t) \cdot v(t) = 0$.

7. Find the tangential and normal components of the acceleration vector of $r(t) = \frac{1}{t}\,\mathbf{i} + \frac{1}{t^2}\,\mathbf{j} + \frac{1}{t^3}\,\mathbf{k}$ at the point $(1, 1, 1)$.
Solution
We have

$$r'(t) = -\frac{1}{t^2}\,\mathbf{i} - \frac{2}{t^3}\,\mathbf{j} - \frac{3}{t^4}\,\mathbf{k}$$

and

$$r''(t) = \frac{2}{t^3}\,\mathbf{i} + \frac{6}{t^4}\,\mathbf{j} + \frac{12}{t^5}\,\mathbf{k}.$$

The point $(1,1,1)$ corresponds to $t = 1$, where $r'(1) = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $r''(1) = 2\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}$, and

$$r'(1) \times r''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -3 \\ 2 & 6 & 12 \end{vmatrix} = \begin{vmatrix} -2 & -3 \\ 6 & 12 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -3 \\ 2 & 12 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -2 \\ 2 & 6 \end{vmatrix} \mathbf{k}$$

$$= (-24 + 18)\,\mathbf{i} - (-12 + 6)\,\mathbf{j} + (-6 + 4)\,\mathbf{k} = -6\mathbf{i} + 6\mathbf{i} - 2\mathbf{k}.$$

So at the point $(1,1,1)$,

$$a_T = \frac{r'(1) \cdot r''(1)}{|r'(1)|} = \frac{-2 - 12 - 36}{\sqrt{1 + 4 + 9}} = \frac{-50}{\sqrt{14}} \text{ and}$$

$$a_N = \frac{|r'(1) \times r''(1)|}{|r'(1)|} = \frac{\sqrt{36 + 36 + 4}}{\sqrt{14}} = \sqrt{\frac{76}{14}} = \sqrt{\frac{38}{7}}.$$

8. Draw a contour map of the function $f(x,y) = \ln(x^2 + 4y^2)$.
Solution
The level curves are $\ln(x^2 + 4y^2) = k$ or $x^2 + 4y^2 = e^k$, which is a family of ellipses: