Homework 5

1. (a) Find the tangent plane approximation to the function $f(x, y) = x^3y - y^2 + 5$ at the point $(x, y) = (1, 1)$. (b) Approximate $f(1.2, 0.9)$ using linear approximation you found in part (a).

2. 3Suppose that $u(x, y) = F(x - 2y) + G(x + 2y)$.

   (a) Compute u_x, u_y

   (b) Compute u_{xx}, u_{yy}

   (c) Find the value of k such that $u_{xx} = ku_{yy}$.

   Solution.

$$\begin{aligned} u_x &= F'(x - 2y) + G'(x - 2y) \\ u_y &= -2F'(x - 2y) - 2G'(x - 2y) \end{aligned}$$

$$\begin{aligned} u_{xx} &= F''(x - 2y) + G''(x - 2y) \\ u_{yy} &= 4F''(x - 2y) + 4G''(x - 2y) \end{aligned}$$

   So then $u_{yy} = 4u_{xx}$ i.e. $k = 1/4$.

3. The wind-chill index is modeled by the function

$$W = 13 + 0.6T - 11v^{0.15} + 0.4Tv^{0.15}$$

   where T is the temperature and v is the wind speed. When $T = -15^0$C and $v = 30$ km/h, by how much would you expect the apparent temperature to drop if (a) the actual temperature decreases by 2^0C? (b) the wind speed increases by 2 km/h?

   Solution. (a) $W_T = 0.6 + 0.4v^{0.15}$ so that $W_T(-15, 30) = 1.266$. The wind-chill index would decrease by $1.266 \times 2 = 2.53$ degrees when the temperature decreases by 2 degree.

   (b) Similarly $W_v = -1.65v^{-0.85} + 0.06Tv^{-0.85}$ and $W_v(-15, 30) = -0.14$ so that the wind-chill temperature *decreases* by $0.14 \times 2 = 0.28$ degrees when the wind *increases* by 2 km/h.

4. You are given a function with the property that $f_x(x, y) = 2x + ayx^2$ and $f_y = x^3 + 1$. Determine the constant a. Hint: use equality of mixed partial derivatives.

   Solution. Compute $f_{xy} = ax^2$ and compute $f_{yx} = 3x^2$. Equating the two yields $a = 3$.

5. Given a surface defined implicitly through

$$x^2 - y^2 + z^2 + xy^3z = 2.$$

   (a) Determine $\partial z/\partial x$ at the point $(x, y, z) = (1, 1, 1)$.

   (b) Determine $\partial z/\partial y$ at the point $(x, y, z) = (1, 1, 1)$.

(c) Determine the equation of the plane tanget to this surface at the point $(x, y, z) = (1, 1, 1)$.

Solution. (a) Differentiate implicitly:

$$2x + 2z\frac{\partial z}{\partial x} + y^3 z + xy^3\frac{\partial z}{\partial x} = 0$$

Plug in $x, y, z = 1$ to get

$$2 + 2\frac{\partial z}{\partial x} + 1 + \frac{\partial z}{\partial x} = 0$$

so that

$$\frac{\partial z}{\partial x} = -1.$$

(b) Differentiate implicitly:

$$-2y + 2z\frac{\partial z}{\partial y} + 3y^2 xz + xy^3\frac{\partial z}{\partial y} = 0.$$

Plug in $x, y, z = 1$ to get

$$-2 + 2\frac{\partial z}{\partial y} + 3 + \frac{\partial z}{\partial y} = 0.$$

so that

$$\frac{\partial z}{\partial x} = -1/3.$$

(c)

$$z - 1 = -1\,(x - 1) + (-1/3)\,(y - 1)$$

or

$$z + x + y/3 = 7/3$$

6. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = h(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(a) On the $t - v$ plane, make a rough sketch of several well-chosen contours of $h(v,t)$. Indicate the values of contours chosen. Hint: you can print out the table above and draw the contours directly on the table.

(b) Estimate the values of $h_v(30,20)$ and $h_t(30,20)$. What is the practical interpretation of these values?

(c) Using part (c) and linear approximation, estimate $h(33,25)$ as best as you can.

Solution. (a)

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(b) From the table we can use any of the following estimations:

$$h_v(30,20) \approx \frac{17-8}{10} \approx \frac{28-17}{10} \approx \frac{28-8}{20} \approx 1$$

This means the height grows by one unit per one knot increase in wind speed. Similarly,

$$h_t(30,20) \approx \frac{17-16}{5} \approx \frac{18-17}{5} \approx \frac{18-16}{10} \approx 0.2$$

So the wind height grows by 0.2 units with every passing hour.

(d) Using linear approximation,

$$\begin{aligned} h(33,25) &\approx h(30,20) + 3h_v(30,20) + 5h_t(30,20) \\ &= 17 + 3 \times 1 + 5 \times 0.2 = 21. \end{aligned}$$

7. Let $u(x,t)$ be the temperature of a metal bar at a one-dimensional location x and time t. Then $u(x,t)$ satisfies the *heat equation* which has the following non-dimensional form:

$$u_t = u_{xx}$$

(a) There is a solution of the Heat equation of form $u(x,t) = e^{-\lambda t}\sin(mx)$. Determine a relationship between λ and m.

(b) Find *all* solutions of the heat equation that have the form $u(x,t) = F(t)\sin(mx)$ for some function $F(t)$. Show that they are all some scalar multiple of the solution you found in part (a).

(c) Can you find any other solutions of the heat equation that are *not* of the form $u(x,t) = F(t)\sin(mx)$?

Solution. (a) $\lambda = m^2$.

(b) We get $F'(t) = -m^2 F(t)$ whose solution is $F(t) = Ae^{-m^2 t}$ where A is arbitrary constant so that so the most general solution of this form is $u(x,t) = Ae^{-m^2 t}\sin(mx)$.

(c) Some other solutions not of this form include:

$$\begin{aligned} u(x,t) &= e^{-t}\sin(x) + e^{-4t}\sin(2x); \\ u(x,t) &= Ax + B; \\ u(x,t) &= \exp(m^2 t)\exp(mx) \end{aligned}$$

and there are many others...

8. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.3T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 10 L to 12 L and the temperature decreases from 310 K to 305 K.

   Solution. We have
   $$dPV + dVP = 8.3dT$$
   with $V = 10,\ dV = 2,\ T = 310,\ dt = -5,\ P = 8.3T/V = 257.3$ which gives
   $$dP \times 10 + 2 \times 257.3 = 8.3(-5)$$
   so that
   $$dP = -55.61.$$

9. The temperature at a point (x,y) is given (in degrees Celsius) by $T(x,y)$. A bug crawls so that its position after t seconds is given by $x = t^2,\ y = 1 + t^3$. Suppose that $T_x(1,2) = 1$ and $T_y(1,2) = 3$. How fast is the temperature rising on the bug's path at $t = 1$?

   Solution. The temperature along the bug's path is given by $f(t) = T(x(t), y(T) = T(t^2, 1 + t^3)$. Then $f'(t) = T_x \frac{dx}{dt} + T_y \frac{dy}{dt} = 1 \times 2 + 3 \times 3 = 11$.