Homework 6

1. (a) Use chain rule to find the partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ of $z = e^{x^2 y}$, where $x(u,v) = \sqrt{uv}$ and $y(u,v) = \frac{1}{v}$.

   (b) Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ only in terms of u and v and evaluate at $(u,v) = (1,1)$.

   Solution:

   (a)

   $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2xye^{x^2 y})(\frac{v}{2\sqrt{uv}}) + (x^2 e^{x^2 y})(0) = xye^{x^2 y}\frac{v}{\sqrt{uv}}$$

   and

   $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2xye^{x^2 y})(\frac{v}{2\sqrt{uv}}) + (x^2 e^{x^2 y})(-\frac{1}{v^2}) = e^{x^2 y}(xy\frac{u}{\sqrt{uv}} - x^2\frac{1}{v^2})$$

   (b)

   $$\frac{\partial z}{\partial u} = \sqrt{uv}\frac{1}{v}e^{uv\frac{1}{v}}\frac{v}{\sqrt{uv}} = e^u$$

   $$\frac{\partial z}{\partial v} = e^{uv\frac{1}{v}}(\sqrt{uv}\frac{1}{v}\frac{u}{\sqrt{uv}} - uv\frac{1}{v^2}) = e^u(\frac{u}{v} - \frac{u}{v}) = 0$$

   So

   $$\frac{\partial z}{\partial u}\Big|_{(u,v)=(1,1)} = e \text{ and } \frac{\partial z}{\partial v}\Big|_{(u,v)=(1,1)} = 0$$

2. (a) Find the directional derivative of $f(x,y,z) = xy^2z^3$ at $P(2,1,1)$ in the direction of $Q(0,-3,5)$.

   (b) In which direction is the directional derivative maximized and how much is it?

   Solution:

   (a) The unit vector in the direction of $\vec{PQ} = \langle 0-2, -3-1, 5-1\rangle = \langle -2,-4,4\rangle$ is

   $$u = \frac{1}{\sqrt{4+16+16}}\langle -2,-4,4\rangle = \frac{1}{6}\langle -2,-4,4\rangle.$$

Also

$$\nabla f = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle,$$

so $\nabla f(2,1,1) = \langle 1,4,6 \rangle$. So the directional derivative is $D_u f(2,1,1) = \nabla f(2,1,1) \cdot u = \langle 1,4,6 \rangle \cdot \frac{1}{6}\langle -2,-4,4 \rangle = \frac{1}{6}(-2-16+24) = 1$.

(b) The directional derivative is maximed in the direction of the gradient vector $\nabla f(2,1,1) = \langle 1,4,6 \rangle$ and it is $|\nabla f(2,1,1)| = |\langle 1,4,6 \rangle| = \sqrt{1+16+36} = \sqrt{53}$.

3. Consider the surface $xy + yz + zx = 5$. Find (a) the tangent plane at $(1,2,1)$ and (b) the normal line at $(1,2,1)$.
Solution:

(a) Consider $F(x,y,z) = xy + yz + zx$. Its gradient is $\nabla F(x,y,z) = \langle y+z, x+z, x+y \rangle$, so $\nabla F(1,2,1) = \langle 3,2,3 \rangle$, which is a normal vector to the tangent plane at $(1,2,1)$. So the tangent plane is given by

$$\begin{aligned} 3(x-1) + 2(y-2) + 3(z-1) &= 0 \\ \text{or } 3x + 2y + 3z &= 10 \end{aligned}$$

(b) The normal line has direction $\langle 3,2,3 \rangle$, so it has the parametric equations

$$\begin{cases} x = 1 + 3t \\ y = 2 + 2t \\ z = 1 + 3t \end{cases}$$

4. Find all critical points of the function

$$f(x,y) = x^3 + y^2 - 2xy + x - 2y.$$

Use the second derivative test to classify the critical points as either min, max or a saddle point. If it is a min or max, is it a global min or max?
Solution:
The first partial derivatives of the function are

$$f_x(x,y) = 3x^2 - 2y + 1 \text{ and } f_y(x,y) = 2y - 2x - 2,$$

so in order to find the critical points we have to solve the following system:

$$\begin{cases} 3x^2 - 2y + 1 = 0 \\ 2y - 2x - 2 = 0 \end{cases}$$

From the second equation we take $y = x + 1$ and then from the first we take

$$3x^2 - 2(x+1) + 1 = 0 \text{ or}$$

$$3x^2 - 2x - 1 = 0$$

which gives solutions $x = 1$ and $x = -\frac{1}{3}$. If $x = 1$ then $y = 1 + 1 = 2$ and if $x = -\frac{1}{3}$ then $y = -\frac{1}{3} + 1 = \frac{2}{3}$. So the two critical points are $(1, 2)$ and $(-\frac{1}{3}, \frac{2}{3})$.
Now in order to apply the second derivative test we have

$$f_{xx}(x, y) = 6x$$

$$f_{yy}(x, y) = 2$$

$$f_{xy}(x, y) = -2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 12x - 4$$

and $D(1, 2) = 8$, $D(-\frac{1}{3}, \frac{2}{3}) = -8$. Then:

	D	fxx	conclusion
(1,2)	+	+	local minimum
(-1/3,2/3)	-		saddle point

The local minimum at $(1, 2)$ is not global since $f(1, 2) = -2$ and we can find values of f that are smaller than that, for example $f(-2, 0) = -10$.

5. Find dimensions of the box without a lid with volume 32cm^3 that has minimal surface area.
Solution:
Let x, y, z be the dimensions of the box. The surface area of the box without the lid is $f(x, y, z) = xy + 2xz + 2yz$. We want the volume to be 32cm^3, i.e. $xyz = 32$, or $z = \frac{32}{xy}$. So the function that we want to minimize is

$$f(x, y) = xy + 2x\frac{32}{xy} + 2y\frac{32}{xy} = xy + \frac{64}{y} + \frac{64}{x}.$$

Then we need to solve the system

$$f_x = y - \frac{64}{x^2} = 0$$
$$f_y = x - \frac{64}{y^2} = 0$$

from which we take $x^3 = 64$ or $x = 4$ and $y = \frac{64}{4^2} = 4$. Plugging these back in $xyz = 32$ we take $z = 2$. So the dimensions of the box are $(4, 4, 2)$.

6. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region $x^2 + y^2 \leq 16$.
Solution:
To find the critical points we need to solve $\nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle$. This gives $x = 1$ and $y = 0$ which lies in the interior of the given region, $x^2 + y^2 < 16$.
On the boundary $x^2+y^2 = 16$ we have: Consider the function $g(x, y) = x^2 + y^2$, which has gradient $\nabla g \langle 2x, 2y \rangle$. Then we need to solve the following system for all the values of x, y and λ:

$$4x - 4 = \lambda 2x$$
$$6y = \lambda 2y$$
$$x^2 + y^2 = 16$$

From $6y = \lambda 2y$ we take that either $y = 0$ or $\lambda = 3$.
If $y = 0$ then $x^2 = 16$ or $x = \pm 4$.
If $\lambda = 3$ then $4x - 4 = \lambda 2x \implies x = -2$ and $(-2)^2 + y^2 = 16 \implies y^2 = 12 \implies y = \pm 2\sqrt{3}$. Now we calculate the values at all of the above points:

$$f(1, 0) = -7$$
$$f(4, 0) = 11$$
$$f(-4, 0) = 45$$
$$f(-2, 2\sqrt{3}) = 47$$
$$f(-2, -2\sqrt{3}) = 47$$

So the maximum is 47 at either $(-2, 2\sqrt{3})$ or $(-2, -2\sqrt{3})$, and the minimum is -7 at $(1, 0)$.

7. The total production of a certain product is modeled by the Cobb-Douglas function $P = 100L^{3/4}K^{1/4}$, where L represents the units of labor and K represents the units of capital. Each labor unit costs \$200 and each capital unit costs \$250. If the total expenses for labor and capital cannot exceed \$50,000, find the maximum level of production.

Solution:

We want to maximize $P(L, K) = 100L^{\frac{3}{4}}K^{\frac{1}{4}}$ with the constrain of the total cost to be 50,000, i.e.

$$g(L, K) = 200L + 250K = 50,000.$$

We have

$$\nabla P = \langle \frac{3}{4}100L^{-\frac{1}{4}}K^{\frac{1}{4}}, \frac{1}{4}100L^{\frac{3}{4}}K^{-\frac{3}{4}} \rangle = \langle 75L^{-\frac{1}{4}}K^{\frac{1}{4}}, 25L^{\frac{3}{4}}K^{-\frac{3}{4}} \rangle$$

and

$$\lambda \nabla g = \langle \lambda 200, \lambda 250 \rangle.$$

We need to solve the system

$$\begin{aligned} 75L^{-\frac{1}{4}}K^{\frac{1}{4}} &= \lambda 200 \\ 25L^{\frac{3}{4}}K^{-\frac{3}{4}} &= \lambda 250 \\ 200L + 250K &= 50,000 \end{aligned}$$

Solving for λ the first two we have

$$\begin{aligned} \lambda &= \frac{75}{200}(\frac{K}{L})^{\frac{1}{4}} \text{ and} \\ \lambda &= \frac{1}{10}(\frac{L}{K})^{\frac{3}{4}}, \end{aligned}$$

so setting them equal gives

$$\frac{75}{20}(\frac{K}{L})^{\frac{1}{4}} = (\frac{L}{K})^{\frac{3}{4}} \implies \frac{75}{20} = (\frac{L}{K})^{\frac{3}{4}}(\frac{L}{K})^{\frac{1}{4}} \implies \frac{L}{K} = \frac{75}{20} \implies L = \frac{75}{20}K.$$

Now from the third one

$$200\frac{75}{20}K + 250K = 50,000 \implies K = 50$$

$$\text{and } L = \frac{75}{20}50 = 187.5.$$

So the production in this case is $P(50, 187.5) = 13,478.22$.

8. Use Lagrance multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = \frac{p}{2}$ *and* x, y, z *are the lengths of the sides.*
Solution:
Let $f(x, y, z) = s(s-x)(s-y)(s-z)$ and $g(x, y, z) = x + y + z$ for the perimeter of the triangle. Then we want to maximize f with the constrain $x + y + z = p$. We have

$$\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y) \rangle$$

$$\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$$

So we want to solve the system

$$\begin{aligned} -s(s-y)(s-z) &= \lambda \\ -s(s-x)(s-z) &= \lambda \\ -s(s-x)(s-y) &= \lambda. \end{aligned}$$

Or, by eliminating λ, the system

$$\begin{aligned} (s-y)(s-z) &= (s-x)(s-z) \\ (s-x)(s-z) &= (s-x)(s-y). \end{aligned}$$

The latter gives $x = y$ and $y = z$, so $x = y = z = \frac{p}{3}$, which means that the triangle is equilateral.

9. The plane $2x + 2y + z = 2$ intersects the surface $z = x^2 + y^2$. Use Lagrance multipliers to:

   (a) Find the point of intersection of these two surfaces which is closest to the z-axis.

   (b) Find the point of intersection which is furthest away from the z-axis.

   Solution:
   We want ot minimize and maximize the distance from the z-axis which

is $\sqrt{x^2+y^2}$. Instead, we will minimize/maximize its square $f(x,y,z) = x^2+y^2$. The constrains are

$$g(x,y,z) = 2x+2y+z = 2 \text{ and}$$

$$h(x,y,z) = x^2+y^2-z = 0.$$

Then

$$\begin{aligned} \nabla f &= \langle 2x, 2y, 0\rangle \\ \lambda\nabla g &= \langle 2\lambda, 2\lambda, \lambda\rangle \\ \mu\nabla h &= \langle 2\mu x, 2\mu y, -\mu\rangle \end{aligned}$$

So we need to solve the following system:

$$\begin{aligned} \nabla f &= \lambda\nabla g + \mu\nabla h \\ 2x+2y+z &= 2 \\ x^2+y^2-z &= 0. \end{aligned}$$

From the first one we have

$$\begin{cases} 2x = 2\lambda + 2\mu x \\ 2y = 2\lambda + 2\mu y \\ \lambda = \mu \end{cases} \implies \begin{cases} x = \lambda(1+x) \\ y = \lambda(1+y) \end{cases} \implies \frac{x}{1+x} = \frac{y}{1+y} \implies x = y$$

Now from the last two we take $z = 2-4x$ and $z = 2x^2$, and by eliminating z we take $x = -1\pm\sqrt{2}$. So finally we have two solutions $(-1-\sqrt{2}, -1-\sqrt{2}, 6+4\sqrt{2})$ and $(-1+\sqrt{2}, -1+\sqrt{2}, 6-4\sqrt{2})$. Then

$$f(-1-\sqrt{2}, -1-\sqrt{2}, 6+4\sqrt{2}) = 2(1+\sqrt{2})^2 \text{ and}$$

$$f(-1+\sqrt{2}, -1+\sqrt{2}, 6-4\sqrt{2}) = 2(1-\sqrt{2})^2$$

which shows that the first point is the maximum of the f and the last one is the minimum.