MATH 2300, Homework 5

1. Recall that the Poisson distribution is defined as

$$P_v(n) = \frac{v^n}{n!} \exp(-v),$$

where $v = \lambda t$, and it represents the probability that n events occur within a time interval of length t for an event that follows a Poisson process with a rate of λ.

   (a) Show that $P_v(n)$ is indeed a probability distribution on $n = 0, 1, \ldots$. That is, show that $\sum_{n=0}^{\infty} P_v(n) = 1$.

   (b) The mean is defined as $\mu = \sum_{n=0}^{\infty} n P_v(n)$. Show that $\mu = v$.

   (c) The variance is defined as $\sigma^2 = \sum_{n=0}^{\infty} (n - \mu)^2 P_v = \left(\sum_{n=0}^{\infty} n^2 P_v\right) - \mu^2$ where μ is the mean found in part (b). Show that $\sigma^2 = v$.

   (d) Let $v = 20$. On the same plot, using Matlab, graph n vs. $P_v(n)$ for $n = 0 \ldots 40$, as well as a Normal distribution with mean and standard deviation σ as found in part (b) and (c) (i.e. $\mu = v$, $\sigma = \sqrt{v}$.). Hand in your plot. Make a conjecture based on what you observe. BONUS: give an argument for why that conjecture might be true.

2. The service counter at Pi Airlines has a single queue for waiting customers and two ticket agents. One of the agents is on duty at all times; the other agent goes on duty whenever the queue of customers becomes too long. Suppose that customer arrivals to the counter are well modelled as a Poisson process with rate 45/hour. The agents both work at rate of 30 customers per hour, and the second agent goes on duty if there are 3 or more customers at the counter (including the ones being served). Service times are modeled as exponentially distributed.

   (a) Write a Matlab program to simulate this model. You can modify sample code from class for this; use $dt = 0.01$ and run 100000 time steps. From your simulation, compute the long-time probabilities π_n that the queue has size exactly n, and plot π_n as a function of n. What is the numerical value that you get for π_0?.

   (b) Determine π_n using queuing theory. Compare the value of π_0 with that of the numerical simulation you did in part (a). Plot π_n as a function of n with $n = 0 \ldots 25$ and compare with the plot in part (a). You may wish to superimpose the two plots together as in the sample code. Hand in your plot.

   (c) Determine the fraction of time that the second agent is on duty. *Note: you can use Matlab to help answer the questions (c-f) below.*

   (d) Determine the average length of the queue, including those being served.

   (e) Determine the average length of the queue, *not* including those being served.

   (f) Now treat the number of customers that trigger the second agent to go on duty as a decision variable. When should the second agent go on duty to keep the average length of the queue, *not* including those being served, under 5?

3. A restaurant is currently operating a single drive-through window. When one person is working, it takes on average 2 minutes/customer to fill the order. With two people working, this can be reduced to 1 minute and 15 seconds per customer. Cars arrive at a rate of 24 per hour.

   (a) Determine the average waiting time for one and two-person systems (note: in either case, there is only a single drive-through window, so it's a queue with one server only).

   (b) Suppose that when there are six cars waiting for the drive-through window, then any additional arrivals cannot join the queue. In other words, the length of the drive-through lane is limited, so that when there are six cars there, then any new arrivals are lost to the restaurant. Given this information, evaluate the one - and two-person systems in terms of their affect on the rate of cars lost to the drive-through window.

4. In Canada, most restaurants follow "every table has is own server" model whereas in many other places such as China, the model is "every table has multiple servers". Which model is better? To evaluate the differing models contrast the following two scenarios.

   Scenario A (Canada): Customers arrive to a restaurant at a rate of 1 per minute. There are 3 servers and each can serve at a rate of one customer every two minutes. But each customer can be served by one server only.

   Scenario B (China): Same rates as scenario A, except that multple servers can serve the same customer when they are available. The time it takes to serve decreases in proportion to the number of servers being used (so assume that two servers working together can serve one customer in one minute and if all three are available then it takes them only 40 seconds)

   Compute the probability distributions of queue lengths under both scenarios and determine the average queue lengths to serve as well as the average waiting times. Which model do you think is better?

5. *Death process (p 359).* Consider a *pure death process.* In such a process, individuals persist only until they die and there are no replacements. The assumptions are similar to those in the pure birth process, but now each individual, if still alive at time t, is removed in $(t, t+\Delta t)$ with probability $\mu\Delta t$. Suppose that initially, there are n_0 individuals at time $t=0$, and let $P_n(t)$ be the probability that n individuals are alive at time at time t.

   (a) Derive the system of differential equations for $P_n(t)$.

   (b) For simpilciity, we will assume here and below that $\mu = 1$. Show that $P_{n_0}(t) = e^{-n_0 t}$, and that $P_{n_0-1} = n_0 e^{-(n_0-1)t}\left(1-e^{-t}\right)$.

   (c) In general, show that that

   $$P_k(t) = \begin{pmatrix} n_0 \\ k \end{pmatrix} e^{-kt}\left(1-e^{-t}\right)^{n_0-k}, \quad k = n_0, n_0-1, \ldots 0 \tag{1}$$

   satisfies the system of differential equations for $P_k(t)$. Now argue that this formula can be also be shown by using Poisson process plus binomial distribution [without solving any differential equations!]

   (d) Use part (c) to show that the probability that the population is extinct at time t is given by $\left(1-e^{-t}\right)^{n_0}$.

   (e) **Simulation:** Here is a some matlab code:

   ```
   dt=0.001;
   t=0:dt:2;
   n0=15;
   P=t*0; % population at time t
   P(1)=n0;
   for i=2:numel(P)
   n=P(i-1);
   death = rand()<dt*n;
   if death, n=n-1; end;
   P(i)=n;
   end;
   plot(t,P);
   ```

   It simulates the death process with $n_0 = 20$, $\mu = 1$, using $dt = 0.001$ and runs till $t = 2$.

   Now modify this code to run this simulation 1000 times, each time recording the how many individuals survive until $t = 2$. Store these numbers in an array. From this, compute the probability of extinction (i.e. what fraction of simulations yields zero individuals at the end?). Report what you get. Does it agree with part (d)?

(f) From the data you computed in question (e), compute your estimate for $P_k(2)$ with $k = 0 \ldots 20$. Then on the same plot, graph your estimate and the analytical result (1). Comment on what you observe.