Final exam preparation

- Go over all the homework, sample, and midterm questions.
- Attempt all the questions in this handout.
- There will be no calculators or any other aids allowed for the final exam.

Final exam topics

1. Root finding (chap 1, chap 2.7.1): bisection method – fixed point iterations – Newton's method – secant method – estimating error...
2. Interpolation (chap 3): Lagrange interpolation formula, Newton's differences formula, error estimates, Chebychev polynomials...
3. Integration/differetiation (chap 5): finite differences; integration: basic formulas, Romberg, Legendre polynomails; Gaussian quadrature, adaptive quadrature, error estimates...
4. ODE's (chap 6, chap 7.1): basic methods; local/global truncation error; variable step-size methods; multi-step methods; stability.
5. Data fitting (chap 4): overdetermined systems; least squares; nonlinear fitting, Gauss-Newton method.

Additional sample questions

1. A bisection method The bisection method is applied to the function $y = (x-1)(x-3)(x-4)$. The initial interval used for the method is $[a, b] = [0, 5]$.
   (a) Compute the next two iterates. Which root will the method converge to, and why?
   (b) How many iterations are needed until the root is found to within 10^{-3} accuracy?
   (c) Use Secant method to compute x_3 if $x_1 = 2$ and $x_2 = 2.5$.
2. Set up Newton's method to determine the point x and the number m for which the line $y = mx$ intersects the curve $y = \cos(x)$ tangentially: that is $y = mx$ is tangent to $y = \cos(x)$ at the point of intersection x.
3. Find the polynomial $P(x)$ which interpolates $\sin x$ at $x = 0, \pi/2, \pi$. Then find a number E such that $\left|P(\frac{\pi}{4}) - \sin(\pi/4)\right| \leq E$.
4. A Chebychev polynomial T_n on $[-1, 1]$ has the form $T_n(x) = 2^{n-1}(x - x_1)\ldots(x - x_n)$ where $x_1 \ldots x_n$ are its roots, $x_i = \cos\frac{(2i-1)\pi}{2n}$. Moreover it has the property that $|T_n(x)| \leq 1$ for $x \in [-1, 1]$.
   (a) Suppose that a function $f(x) = \sin(2x)$ is interpolated on $[-1, 1]$ at the roots of Chebychev polynomial $x_1 \ldots x_n$ by a polynomial $P_{n-1}(x)$. Give the bound for the error $E = |P_{n-1}(x) - f(x)|$, with $x \in [-1, 1]$.
   (b) Suppose that a function $f(x) = \sin(2x)$ is interpolated on the interval $[0, 5]$, at n points using Chebychev interpolation (that is, at the points which are the appropriately scaled and translated Chebychev roots) by a polynomial $P_{n-1}(x)$. Give the bound for the error $E = |P_{n-1}(x) - f(x)|$, with $x \in [-1, 1]$.
5. A free cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 2(x-1) - (x-1)^3 & 1 \leq x < 2 \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & 2 \leq x \leq 3 \end{cases}$$

   Find a, b, c, d. Note that a free spline on $[1, 3]$ satisfies $s''(1) = 0 = s''(3)$.

6. Someone has given you a subroutine for approximating the solution of an initial value problem. You don't know the order of the method, so you run the code on a problem with a known solution for two values of h and get the following errors:

h	$\|error\|$
0.01	1.2×10^{-5}
0.005	1.51×10^{-6}

You conclude the order of the method is $O(h^p)$ where p is what integer?

7.

(a) Find the Lagrange polynomial interpolating $y(x_0)$, $y(x_0 + \frac{h}{3})$ and $y(x_0 + h)$. Include the error term.

(b) Use the above Polynomial to find an approximation to $y'(x_0)$ using the values at $x = x_0\,, x_0 + \frac{h}{3}\,, x_0 + h$. Also estimate the error term.

8. The centered differences formula for the second derivative of a function $y(x)$ is

$$y_c''(x) = (y(x+h) + y(x-h) - 2y(x))/h^2.$$

(a) Estimate the error $|y_c''(x) - y''(x)|$ in terms of h.

(b) Use Richardson extrapolation to come up with an approximation to $y''(x)$ which is accurate up to $O(h^4)$.

9. You have a numerical method $N(h)$ which has $O(h)$ error. Applying $N(h)$ for three different values of h you get the following behaviour:

h	0.2	0.1	0.05
$N(h)$	1.6	1.4	1.3

Use Richardson extrapolation to get the best possible estimate you can for $N(0)$.

10. Given that $f(x)$ is sufficiently smooth, find the bound for the error $E = \left|\int_{-1}^{1} f(x)dx - M\right|$ where $M = f(1) + f(-1)$.

11.

(a) It is known that $g(x) = \frac{1}{x^2+1} - 2x + \exp(\sin(x)) - \cos(\exp(x))$. Find a constant M such that $|g(x)| < M$ on the interval $[-1, 2]$.

(b) Let T_n be the Trapezoid rule approximation to $I = \int_{-1}^{2} f(x)dx$. Suppose that $f''(x) = g(x)$ where g is as in part (a). How large should you choose n to guarantee that $\left|\int_{-1}^{2} f(x) - T_n\right| \leq 10^{-3}$? Remark: it is known that $\left|\int_a^b f(x) - T_n\right| \leq \frac{M}{24}(b-a)h^2$ where M is a constant such that $|f''(x)| \leq M$ for all $x \in [a, b]$.

12. Use two point Gaussian quadrature to approximate,

$$\int_0^1 \cos(x^2)\,dx$$

You may use the table below. You do not need to simplify your answer.

Points	Weighting Factors	Function Arguments
2	c1 = 1.000000000	x1 = -0.577350269
	c2 = 1.000000000	x2 = 0.577350269
3	c1 = 0.555555556	x1 = -0.774596669
	c2 = 0.888888889	x2 = 0.000000000
	c3 = 0.555555556	x3 = 0.774596669
4	c1 = 0.347854845	x1 = -0.861136312
	c2 = 0.652145155	x2 = -0.339981044
	c3 = 0.652145155	x3 = 0.339981044
	c4 = 0.347854845	x4 = 0.861136312

13. Consider the following initial value problem

$$y' = y - 2, \quad y(0) = 1\,. \tag{1}$$

   (a) Take one time step of $h = 0.1$ with the first order Euler's method to approximate $y(0.1)$.
   (b) Repeat part (a) with backwards Euler's method.
   (c) Describe the stability region of the Euler's method.
   (d) Show that the backwards Euler's method is stable for all $h > 0$.

14. Given the following class of methods to integrate $y' = f(y)$:

$$y_{i+1} = y_i + h\left(\frac{1}{4}f(y_i) + af(y_i + hbf(y_i))\right). \tag{2}$$

   (a) Determine the values of a, b which will minimize the local error. What will be the local and the global error?
   (b) Apply this method to the ode $y' = -y,\ y(0) = 1$. For which *real* values of $h > 0$ it true that $y_n \to 0$ as $n \to \infty$?

15. Consider the following multi-step method:

$$y_{i+1} = y_i + ay_{i-1} + h\left(bf(y_i) + cf(y_{i-1})\right).$$

   (a) Determine the values of a, b and c which will minimize the local error. What will be the local and the global error?
   (b) Is the method you found in part (a) stable or unstable if h is sufficiently small?

16. Given a system

$$x^3 + y^3 = 1; \quad x^2 = \sin(y).$$

   (a) Set up the multi-variable Newton method to determine the root of this system.
   (b) Given an initial guess $(x_0, y_0) = (0, 0)$, determine the next iteration (x_1, y_1).

17. Write out the linear system for a, b such that the quadratic $y = ax + bx^2$ is the least squares approximation to the following data.

x_i	0	2	3	4
y_i	0	3	3	2

   Note: You do not need to solve this system.

18. Assume we have the data set $(x_1, y_1), \ldots, (x_n, y_n)$. We wish to determine the constants a and b which minimize the sum of the square of the error for the model $y = \frac{ax}{x+b}$.

(a) Give the nonlinear least squares equations which must be satisfied for optimal values a and b. **Note:** Just give the equations which a and b must satisfy. You do not need to consider any iterative method for finding the values.

(b) How can you transform the problem into a linear problem?

(c) Write down the equations that you need to solve for a, b, which minimizes the error for the transformed data using linear least squares.

19. A rocket shoots straight up. After some rescaling, its height satisfies the ODE

$$y''(t) = -\frac{a}{(1+y)^2}.$$

Describe how you would use the Shooting method to determine the value of a so that $y(t)$ satisfies the above equation with additional constraints $y(0) = 0,\ y(1) = 1,\ y'(1) = 0$.