Math 2400 Final exam, January 2011

Please write all your answers in the booklet provided. You have 3 hours. No calculators are allowed.

1. Set up the Newton's method to determine $3^{1/3}$. Starting with an initial guess of $x_0 = 1$, compute x_1.

2.

   (a) Determine a polynomial $p(x)$ of degree 2 such that $p(-1) = 1,\ p(0) = 0$, and $p(1) = 2$.

   (b) Suppose that a function $f(x)$ has the properties that $f(-1) = 1,\ f(0) = 0$, and $f(1) = 2$. Moreover suppose that$|f'''(x)| \leq 5$ for all x. Find two numbers a, b such that $a < p(0.5) - f(0.5) < b$.

3. Consider the iteration $x_{k+1} = f(x_k)$ where $f(x) = (a+1)x - x^2$.

   (a) Verify that this iteration has a fixed point $x = a$.

   (b) Suppose that x_0 is very close to a. For which values of a is it true that $x_k \to a$ as $k \to \infty$?

4. A function $y(x)$ is tabulated as follows:

x	0	1	2
y	2	2	1

   (a) Estimate $\int_0^3 y^2 dx$ using the Trapezoid rule.

   (b) Suppose it is known that $|y'| < \frac{1}{x+1} - \exp(-x)$ and $|y''| < 4\cos\left(x^2\right) - \sqrt{x}$. Estimate the maximum error from part (a). Remark: it is known that $\left|\int_a^b f(x) - T_n\right| \leq \frac{M}{24}(b-a)h^2$, where M is a constant such that $|f''(x)| \leq M$ for all $x \in [a, b]$, and T_n is the Trapezoid rule approximation for $\int_a^b f(x)$ with n subintervals.

   (c) Estimate $\int_0^3 y^2 dx$ using Richardson extrapolation as accurately as you can.

5. Consider the following method to estimate of a derivative of $f(x)$ at $x = 0$:

$$M(h) = \frac{f(h) - f(0)}{h}.$$

   (a) Estimate the error $|f'(0) - M(h)|$ in terms of h.

   (b) Use a one step of Richardson extrapolation to come up with an approximation which used $f(h),\ f(h/2)$ and $f(0)$ and which is accurate up to $O(h^2)$.

6. Consider a multistep method for the ODE $y' = f(y)$:

$$y_{i+1} = y_i + h(af(y_i) + bf(y_{i-1})).$$

   Determine the constants a, b so as to minimize the local error.

7. Given the following method to integrate $y' = f(y)$:

$$y_{i+1} = y_i + h\left(\frac{1}{2}f(y_i) + \frac{1}{2}f(y_i + hf(y_i))\right).$$

   Suppose that this method is used to solve the ODE $y' = -y,\ y(0) = 1$. For which real values $h > 0$ is it true that $y_n \to 0$ as $n \to \infty$?

8. Write out the linear system for a, b such that the quadratic $y = ax + bx^2$ is the least squares approximation to the following data:

x_i	0	1	-1	-1
y_i	0	3	3	2

Solve the resulting linear system for a and b.

9. Set up the Newton's method to solve the system $x^2 + y^3 = x^3 + 2y, \quad \ln(y+1) - \exp(x) = 1.$ Starting with an initial guess $x_0 = 0,\ y_0 = 0$, compute the next iteration x_1, y_1.

10. Let $I = \int_{-1}^{1} |x|\, f(x)dx$ and let $N = A(f(t) + f(-t))$. Determine the constants A, t so as to minimize the error $E = |I - N|$. What is the error for the optimal choice of A, t?