MATH 2400, Homework 2

1. You are asked to find the root of
$$x^3 = x^2 + x + 1$$
   (a) On the same graph, plot the two sides of this equation find bounds for the root. Hand in the resulting graph.

   (b) Use Bisection method to find the root, accurate to 10^{-6}. Make a printout of all the intermediate steps. How many iterations were required?

   (c) Repeat part (b) but using Newton's method.

   (d) Repeat part (b) but using Secant method.

2. The goal of this question is to determine the error behaviour of the Secant method. The secant method is the iteration:
$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$
   The idea is the same as for Newton's method. That is, let us assume that x_0 and x_1 are both very close to the root; then expand in Taylor series and estimate how close is x_2 to the root; then repeat.

   (a) Let $\varepsilon_n = x_n - x$ where x is the root of f, i.e. $f(x) = 0$. Assuming $\varepsilon_0, \varepsilon_1$ are very close to zero and $f(x)$ is a smooth function, expand $f(x)$ in Taylor series to obtain:
$$\varepsilon_2 \approx M\varepsilon_0\varepsilon_1$$
   where M is some number depending on $f'(x)$ and $f''(x)$. What is M? HINT: the computation is the same as we did in class for the Newton's method.

   (b) Assuming $\varepsilon_n, \varepsilon_{n-1}$ are small, we have just like in part (a),
$$\varepsilon_{n+1} \approx M\varepsilon_n\varepsilon_{n-1} \tag{1}$$
   or equivalently,
$$M\varepsilon_{n+1} \approx (M\varepsilon_n)(M\varepsilon_{n-1}). \tag{2}$$
   Now let $f_n = \ln(M\varepsilon_n)$. Show that f_n satisfies
$$f_{n+1} = f_n + f_{n-1}. \tag{3}$$

   (c) To solve (3), make a guess
$$f_n = Cp^n$$
   where C is some constant and where p is to be found. Determine the value of p. [you will find there are two possible choices for p; since you want $\varepsilon_n \to 0$ or $|f_n| \to \infty$, this means you'll need to choose p such that $|p| > 1$.].

   (d) Conclude that
$$M\varepsilon_{n+1} \sim (M\varepsilon_n)^p \tag{4}$$
   where p is the quantity you found in part (c). How does this compare to the Newton's method?

   (e) Verify (4) numerically, by using the example of Q.1. To do this, you will need to first estimate M, then make a list of ε_n from (d), then compare to (4).