**Supplemental questions for midterm**

1.

(a) Can the bisection method be used to find the zero of the polynomial $y = (x-2)^2$?

(b) Set up Newton's method to determine the point $x$ and the number $m$ for which the line $y = mx$ intersects the curve $y = \cos(x)$ tangentially: that is $y = mx$ is tangent to $y = \cos(x)$ at the point of intersection $x$.

2. We have the following information about a funcition $f(x)$.

- $f(1) = 2$
- The first order divided difference $f[x_1, x_2] = 2$ for **all values** $x_1 \neq x_2$.

What can you say about this function?

3. Given a function $f(x)$, set up the Newton's method iteration in order to find fixed points of $f(x)$, which satisfy $x = f(x)$.

4. The function $f(x)$ is tabulated below.

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 2 |
| $f(x)$ | 2 | 4 | 3 |

(a) Make a table of divided differences and write down the Newton interpolating polynomial $P_2(x)$ for $f(x)$.

(b) Suppose it is known that $|f'''(x)| < 2$. Estimate the error $|P_2(0.5) - f(x)|$.

(c) Use Trapezoid rule to estimate $\int_0^2 f(x)dx$. Estimate the maximum error given that $|f'''(x)| < 2$. Remark: if $T_n$ is the Trapezoid rule with $n$ subintervals, then $\left|T_n - \int_a^b f(x)dx\right| \leq \max|f''(x)| \frac{h^2(b-a)}{12}$.

5. Someone has given you a subroutine for approximating the solution of an initial value problem. You don't know the order of the method, so you run the code on a problem with a known solution for two values of $h$ and get the following errors:

| $h$ | $\|error\|$ |
|---|---|
| 0.01 | $1.2 \times 10^{-5}$ |
| 0.005 | $1.51 \times 10^{-6}$ |

You conclude the order of the method is $O(h^p)$ where $p$ is what integer?

6.

(a) Construct the polynomial of degree 2, $P(x)$, which interpolates the data $\{(0,1), (0.5, 1.649), (1, 2.718)\}$.

(b) Given that the data comes from the function $y = e^x$, bound the truncation error $|P(0.25) - e^{0.25}|$.

(c) What points should we choose for interpolation if we wish to minimize the error bound for $0 < x < 1$?

7.

(a) Find the Lagrange polynomial interpolating $y(x_0)$, $y(x_0 + \frac{h}{3})$ and $y(x_0 + h)$. Include the error term.

(b) Use the above Polynomial to find an approximation to $y'(x_0)$ using the values at $x = x_0\,, x_0 + \frac{h}{3}\,, x_0 + h$. Also estimate the error term.

8. You have a numerical method $N(h)$ which has $O(h)$ error. Applying $N(h)$ for three different values of $h$ you get the following behaviour:

| $h$ | 0.2 | 0.1 | 0.05 |
|---|---|---|---|
| $N(h)$ | 1.6 | 1.4 | 1.3 |

Use Richardson extrapolation to get the best possible estimate you can for $N(0)$.

9. Given that $f(x)$ is sufficiently smooth, find the bound for the error $E = \left|\int_{-1}^{1} f(x)dx - M\right|$ where $M = f(1) + f(-1)$.

10.

(a) It is known that $g(x) = \frac{1}{x^2+1} - 2x + \exp(\sin(x)) - \cos(\exp(x))$. Find a constant $M$ such that $|g(x)| < M$ on the interval $[-1, 2]$.

(b) Let $T_n$ be the Trapezoid rule approximation to $I = \int_{-1}^{2} f(x)dx$. Suppose that $f''(x) = g(x)$ where $g$ is as in part (a). How large should you choose $n$ to guarantee that $\left|\int_{-1}^{2} f(x) - T_n\right| \leq 10^{-3}$? Remark: it is known that $\left|\int_a^b f(x) - T_n\right| \leq \frac{M}{24}(b-a)h^2$ where $M$ is a constant such that $|f''(x)| \leq M$ for all $x \in [a, b]$.

11. Use two point Gaussian quadrature to approximate,

$$\int_0^1 \cos(x^2)\, dx$$

You may use the table below. You do not need to simplify your answer.

| Points | Weighting Factors | Function Arguments |
|---|---|---|
| 2 | c1 = 1.000000000 | x1 = -0.577350269 |
| | c2 = 1.000000000 | x2 = 0.577350269 |
| 3 | c1 = 0.555555556 | x1 = -0.774596669 |
| | c2 = 0.888888889 | x2 = 0.000000000 |
| | c3 = 0.555555556 | x3 = 0.774596669 |
| 4 | c1 = 0.347854845 | x1 = -0.861136312 |
| | c2 = 0.652145155 | x2 = -0.339981044 |
| | c3 = 0.652145155 | x3 = 0.339981044 |
| | c4 = 0.347854845 | x4 = 0.861136312 |