Romberg integration example

Consider

$$\int_1^2 \frac{1}{x} dx = \ln 2.$$

We will use this integral to illustrate how Romberg integration works. First, compute the trapezoid approximations starting with $n = 2$ and doubling n each time:

$n = 1: \; R_1^0 = \left(1 + \frac{1}{2}\right)\frac{1}{2} = 0.75;$
$n = 2: \; R_2^0 = 0.5\left(\frac{1}{1.5}\right) + \frac{0.5}{2}(1 + \frac{1}{2}) = 0.708333333$
$n = 4: \; R_3^0 = 0.25\left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}\right) + \frac{0.25}{2}(1 + \frac{1}{2}) = 0.69702380952$
$n = 8: \; R_4^0 = 0.69412185037$
$n = 16: R_5^0 = 0.69314718191.$

Next we use the formula:

$$R_k^i = \frac{4^i R_k^{i-1} - R_{k-1}^{i-1}}{4^i - 1}$$

The easiest way is to keep track of computations is to build a table of the form:

$$\begin{array}{lllll}
R_1^0 & & & & \\
R_2^0 & R_2^1 & & & \\
R_3^0 & R_3^1 & R_3^2 & & \\
R_4^0 & R_4^1 & R_4^2 & R_4^3 & \\
R_5^0 & R_5^1 & R_5^2 & R_5^3 & R_5^3
\end{array}$$

Starting with the first column (which we just computed), all other entries can be easily computed. For example starting with R_1^0, R_2^0 we find

$$R_2^1 = \frac{4R_2^0 - R_1^0}{3} = 0.694444$$
$$R_3^1 = \frac{4R_3^0 - R_2^0}{3} = 0.693253; \quad R_3^2 = \frac{16R_3^1 - R_2^1}{15} = 0.69317460$$

and so on. Every entry depends only on its left and left-top neighbour. Continuing in this way, we get the following table:

0.75000000000				
0.70833333333	**0.69**444444444			
0.69702380952	**0.693**25396825	**0.6931**7460317		
0.69412185037	**0.6931**5453065	**0.693147**90148	**0.693147**47764	
0.69339120220	**0.693147**65281	**0.6931471**9429	**0.69314718**307	**0.69314718**191

The correct digits are shown in bold (the exact answer to 15 digits is given by $\ln 2 = 0.693147180559945$). Here is the table listing error $R_i^k - \ln 2$.

5.7e-02				
1.5e-02	1.3e-03			
3.9e-03	1.1e-04	2.7e-05		
9.7e-04	7.4e-06	7.2e-07	3.0e-07	
2.4e-04	4.7e-07	1.4e-08	2.5e-09	1.4e-09

Note that each successive iteration yields around two extra digits (*why*?). The final iteration only required $n = 16$ function evaluations, plus $O(\ln n)$ arithmetic operations to build the table.

Exercise. Use four iterations of Romberg integration to estimate $\pi = \int_0^1 \frac{4}{1+x^2} dx$. Comment on the accuracy of your result.