1. Recall from class that the sine Fourier series of $x = 1$ on the interval $(0, \pi)$. Using this series, find the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

   (b) BONUS: Can you find the sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$? (use any method you like).

2. Find the Fourier cosine series of the function $|\sin x|$ on the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

3. Find the solution to the PDE

$$\begin{cases} u_{tt} = u_{xx} \\ u_x(0,t) = 0, \quad u_x(\pi, t) = 0, \\ u(x, 0) = 0 \\ u_t(x, 0) = \cos^2 x. \end{cases}$$

   With the help of computer or otherwise, sketch the solution for several well-chosen values of t.

4. (a) Find the solution to the PDE

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 1, \quad u(\pi, t) = 0, \\ u(x, 0) = 0. \end{cases}$$

   Hint: make an shift to homogenize the boundary conditions...

   (b) With the help of computer or otherwise, sketch the solution for several well-chosen values of t.

   (c) Redo part (a) and (b), but replacing $u(0,t) = 1$ by $u_x(\pi, t) = -1$

5. Use an appropriate Fourier series to solve the problem

$$\begin{cases} u_t = u_{xx} + 1 \\ u_x(0,t) = 0, \quad u(\pi, t) = 0, \\ u(x, 0) = 0. \end{cases}$$

   (a) Determine the solution profile in the limit $t \to \infty$.

   (b) Using either a computer or by hand, sketch the solution for several well-chosen representative values of t.

6. Consider a unit sphere in 3D with an initially homogeneous temperature distribution $u = 1$. It is then plunged into a cold bath with an outside temperature of 0. This situation is modelled by the following PDE system:

$$\begin{cases} u_t = u_{rr} + \frac{2}{r} u_r, \quad 0 < r \leq 1 \\ u_r(0,t) = 0, \quad u(1,t) = 0 \\ u(r, 0) = 1. \end{cases} \tag{1}$$

   (a) Use the separation of variables to write the solution to (1) as a series of the from

$$u = \sum T_n(t) X_n(r)$$

   where X_n satisfies the ODE

$$X_{rr} + \frac{2}{r} X_r + \lambda X = 0$$

   for some eigenvalue λ. What boundary conditions must X_n satsify?

   (b) Solve the resulting eigenvalue problem from $X_n(r)$. Hint: Try the following anzatz: $X = \frac{1}{r} \sin(ar)$ for some constant a.

(c) Write down the full series solution to (1).

7. Let $f(x) = x^2$. Consider four different representations of $f(x)$:

   (1) $f_1(x) = \sum b_n \sin(nx), \, , \quad x \in (0, \pi);$

   (2) $f_2(x) = \sum a_n \cos(nx), \quad x \in (0, \pi);$

   (3) $f_3(x) = \sum a_n \cos(nx) + \sum b_n \cos(nx), \quad , \quad x \in (-\pi, \pi);$

   (4) $f_4(x) = \sum b_n \sin(\left(n + \frac{1}{2}\right) x), \quad x \in (0, \pi).$

   Without computing the coefficients, draw how $f_i(x), i = 1..4$ looks like when extended on the whole real line.