Polynomial and transcendental equations

Consider $p(x) = x^2 + \varepsilon x - 1$, $\varepsilon \to 0$.
What are the roots of p?

- If $\varepsilon = 0$ then $p(x) = x^2 - 1 = 0 \Rightarrow x = \pm 1$.
- If ε is <u>small</u> we can <u>expand</u>:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^2 + \varepsilon(x_0 + \varepsilon x_1 + \dots) - 1 = 0$$

Collect terms in ε:

$O(1)$: $\quad x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$

$O(\varepsilon)$: $\quad 2x_1x_0 + x_0 = 0 \Rightarrow x_1 = -\frac{1}{2}$

$O(\varepsilon^2)$: $\quad 2x_2x_0 + x_1^2 + x_1 = 0 \Rightarrow x_2 = \pm\frac{1}{8}$

$$\Rightarrow x = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

<u>Notation</u>:

- $f(x) = O(g(x))$ as $x \to x_0$ if $\exists\, C, \delta$ s.t. $\left|\frac{f(x)}{g(x)}\right| < C$ whenever $|x - x_0| < \delta$

  <u>Ex</u>: $\sin(x) = O(1)$, $x^3 = O(x^2)$ as $x \to 0$
  $= O(x^3)$

  $2e^x + x = O(e^x)$ as $x \to \infty$

- $f(x) = o(g(x))$ if $\frac{f}{g} \to 0$ as $x \to x_0$

  <u>Ex</u>: $x^3 = o(x^2)$ as $x \to 0$

  $\sin x = x + o(x)$ as $x \to 0$
  $= x + O(x^3)$

- $f(x) \sim g(x)$ as $x \to x_0$ if $\frac{f}{g} \to 1$ as $x \to x_0$
- $f(x) \ll g(x)$ as $x \to x_0$ if $f(x) = o(g(x))$ as $x \to x_0$
- $f(x) \gg g(x)$ as $x \to x_0$ if $g(x) = o(f(x))$

Ex:
- $\varepsilon \ll 1$ means ε is small
- If $\varepsilon \ll 1$ then $\left(\ln \frac{1}{\varepsilon}\right)^{-1} \ll 1$
- $\varepsilon^2 \ll \varepsilon$

Multiple roots: Find roots of

$$p(x) = x^2 - (2+\varepsilon)x + 1, \quad \varepsilon \ll 1.$$

- Note: if $\varepsilon = 0$ then $x^2 - 2x + 1 = (x-1)^2 = 0 \Rightarrow$ $x=1$ is a double root.
- Naïvely, let's try an expansion of the form:

$$x = 1 + \varepsilon x_1 + \dots$$

$$\Rightarrow 2x_1 - 2x_1 - 1 = 0 \Rightarrow 1 = 0$$

- Try: $x = 1 + \varepsilon^p x_1$ where p is to be determined

Then $(1+\varepsilon^p x_1)^2 - (2+\varepsilon)(1+\varepsilon^p x_1) + 1 = 0$

$\Rightarrow$

$$\cancel{1} + \cancel{2\varepsilon^p x_1} + \varepsilon^{2p} x_1^2 - \cancel{2} - \cancel{2\varepsilon^p x_1} - \varepsilon - \varepsilon^{p+1} x_1 + \cancel{1} = 0$$

$$\boxed{\varepsilon^{2p} x_1^2 - \varepsilon - \varepsilon^{p+1} x_1 = 0}$$

Method of dominant balance:

- Choose p such that at least two of the three terms are of the same order and the other term is of smaller order.
- ~~Pos~~ Orders are: ε^{2p}, ε^{1}, ε^{p+1}

  Possible choices:
  - $2p = p+1 \Rightarrow p=1 \Rightarrow \varepsilon^2, \varepsilon^1, \varepsilon^2$ Dominant $\not\#$
  - $p+1 = 1 \Rightarrow p=0$ $\#$
  - $2p = 1 \Rightarrow p = \frac{1}{2}$ the only possibility here.

$$\Rightarrow (X_1^2 - 1)\,\varepsilon + \varepsilon^{3/2} X_1 = 0$$

$$X_1^2 - 1 = 0$$

$$\Rightarrow \boxed{X_1 = \pm 1}$$

- To find further corrections:

  Change vars: $X = 1 + \varepsilon^{\frac{1}{2}} y$

$$\Rightarrow y^2 - 1 + \varepsilon^{\frac{1}{2}} y = 0$$

$$\Rightarrow y = \pm 1 + \varepsilon^{\frac{1}{2}} y_1 + \dots$$

$$\Rightarrow y_1 = \frac{1}{2} \quad \text{[just as in prev. example]}$$

$$\Rightarrow \quad x = 1 + \varepsilon^{\frac{1}{2}} + \frac{1}{2}\varepsilon + O(\varepsilon^{3/2})$$

$$x = 1 - \varepsilon^{\frac{1}{2}} + \frac{1}{2}\varepsilon + O(\varepsilon^{3/2})$$

Alternative: method of iterations

- Rewrite your expression as $x = f(x; \varepsilon)$
- Make initial guess x_0 and iterate: $x_1 = f(x_0; \varepsilon)$
  $x_2 = f(x_1; \varepsilon)$
  ....

Ex: $x^2 - (2+\varepsilon)x + 1 = 0 \Rightarrow (x-1)^2 = \varepsilon x$

$$x = 1 \pm \varepsilon^{\frac{1}{2}} x^{\frac{1}{2}}$$

~~Initia~~ Take $x_0 = 1$ [corresponding to $\varepsilon = 0$]

Positive root: $x_1 = 1 + \varepsilon^{\frac{1}{2}}$

$x_2 = 1 + \varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}})^{\frac{1}{2}} = 1 + \varepsilon^{\frac{1}{2}} + \frac{1}{2}\varepsilon + \ldots$

[same as before]

Singular perturbation:

Suppose we want to find roots of $p(x) = \varepsilon x^3 - x^2 + 1 = 0$, $\varepsilon \ll 1$.

- If $\varepsilon = 0$ then $p(x) = -x^2 + 1 = 0$ is quadratic
  $\Rightarrow$ two roots $x = \pm 1$
- But if $\varepsilon > 0$ then $p(x)$ is a cubic $\Rightarrow$ 3 roots
  The "missing" root $\to \infty$ as $\varepsilon \to 0$.

So we rescale: $x = \varepsilon^p y$ for p to be found;

$$\varepsilon^{3p+1} y^3 - \varepsilon^{2p} y^2 + 1 = 0$$

Dominant balance:

- $3p + 1 = 2p \Rightarrow p = -1$
- $3p + 1 = 0 \Rightarrow p = -\frac{1}{3}$ ✗
- $2p = 0 \Rightarrow p = 0$ ✗

So the only (possible) scaling is $p = -$

$$\Rightarrow \quad y^3 - y^2 + \varepsilon^2 = 0$$

$\Rightarrow \quad y = O(\varepsilon^2)$ [corresponding to $x = \pm 1$]

<u>or</u> $\quad y = 1 + O(\varepsilon^2)$

So to leading order, the three roots are:

$$x \approx \pm 1, \qquad x \sim \varepsilon^{-1} \quad \text{as } \varepsilon \to 0.$$

Transendental equations

<u>Ex</u>: Estimate roots of $x e^{-x} = \varepsilon$ as $\varepsilon \to 0$.

From calculus, $\exists$ two roots, one ~~large~~ large, another one small.

<u>Small root</u>: <u>Expand</u> $\quad x = \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$$\Rightarrow \quad (x_1 + \varepsilon x_2)\left(1 - \varepsilon x_1 - \varepsilon^2 x_2 - \frac{\varepsilon^2}{2} x_1^2 \dots\right) = 1$$

$$\Rightarrow \quad x_1 = 1, \quad -x_1^2 + x_2 = 0, \quad x_2 = 1$$

$$\Rightarrow \quad x \sim \varepsilon + \varepsilon^2 + O(\varepsilon^3)$$

Or using <u>iterations</u>: $\quad x = \varepsilon e^{x}, \quad x_0 = \varepsilon,$

$$x_1 = \varepsilon(1 + \varepsilon + \dots)$$
$$= \varepsilon + \varepsilon^2 + \dots$$

Large root:

$$\ln x - x = \ln \varepsilon$$

$$x = \ln x + \ln \tfrac{1}{\varepsilon}$$

Iterate:

$$x_0 = \ln \tfrac{1}{\varepsilon}$$

$$x_1 = \ln \ln \tfrac{1}{\varepsilon} + \ln \tfrac{1}{\varepsilon}$$

Let $L_1 = \ln \frac{1}{\varepsilon}$, $L_2 = \ln \ln \frac{1}{\varepsilon}$; $\boxed{L_1 \gg L_2}$

$$x_1 = L_1 + L_2$$

$$x_2 = \ln(L_1 + L_2) + L_1$$

$$= \ln\left(L_1\left(1 + \frac{L_2}{L_1}\right)\right) + L_1$$

$$= L_2 + \ln\left(1 + \frac{L_2}{L_1}\right) + L_1$$

$$\boxed{x_2 = L_1 + L_2 + \frac{L_2}{L_1}}$$

Error, in %:

ε	L_1	$L_1 + L_2$	$L_1 + L_2 + \frac{L_2}{L_1}$
0.1	36%	12%	2%
0.001	24%	3%	0.02 %
0.00001	19%	1 %	0.04 %

Ill - conditioning:

Find roots of $p(x) = p_0 + \varepsilon p_1$ where

$$p_0 = (x-1)(x-2)\cdots(x-N)$$

$$p_1 = x^N; \quad \text{i.e.} \quad p(x) = x^n(1+\varepsilon) + \cdots\cdots$$

Let $x = n + \varepsilon x_1 + O(\varepsilon^2)$ be the n-th root of p;

then $$x_1 = -\frac{p_1(x)}{p_0'(x)} = (-1)^{N-n+1} \frac{n^N}{(n-1)!(N-n)!}$$

Take $N = 20$; then

$|x_1|$ attains max when

$n = 16, \quad x_1 = -3.8 \times 10^{10}$

So for expansion $x = n + \varepsilon x_1$ to be accurate,

we need to take $\varepsilon \ll 10^{-10}$ [otherwise $x_1 \varepsilon \geqslant O(1)$ is of the same order as n]

In general, $|x_1|$ attains max ~~at~~ when $n \approx 0.72N$

with $\ln|x_1| \sim 1.28N$

So ε must be <u>extremely</u> small;

$$\varepsilon \ll 10^{-0.55N}$$

This is below machine precision for a typical fixed-precision computer with $N \sim 20$!