Matched asymptotics and singular perturbations ①

Consider: $\begin{cases} \varepsilon u'' + u' = -e^{-x}, & x \in [0,1] \\ u(0) = 0, \quad u(1) = 1 \end{cases}$

In the limit $\varepsilon \to 0$, if we naively drop $O(\varepsilon)$ term, we get: $u' = -e^{-x}$, $u = e^{-x} + C$.

Imposing $u(1) = 1$ we get $u(x) = e^{-x} - e^{-1} + 1$

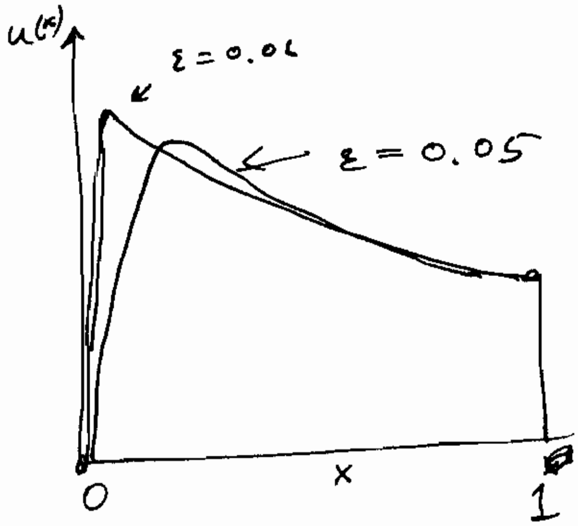
But then "$u(0)$" $= 1 - e^{-1} \neq 0$

This suggests that there is a <u>boundary layer</u> near $x \sim 0$. In fact, exact solution is given by

$$u = \frac{e^{-x}}{1-\varepsilon} + A e^{-x/\varepsilon} + B,$$

$$A = -\frac{(2 - \varepsilon - e^{-1})}{(1-\varepsilon)(1 - e^{-\frac{1}{\varepsilon}})}$$

$$B = \frac{1 - e^{-1} - \varepsilon + e^{-\frac{1}{\varepsilon}}}{(1-\varepsilon)(1 - e^{-\frac{1}{\varepsilon}})}$$



Inside a boundary layer, u has a sharp gradient and $\varepsilon u''$ cannot be ignored.

To resolve the boundary layer, we rescale:

$$\begin{cases} x = \varepsilon^p y & \text{for } p \text{ to be determined} \\ u = \mathcal{U}(y). & \text{We obtain:} \end{cases}$$

$$\varepsilon^{-2p+1}\mathcal{U}_{yy} + \varepsilon^{-p}\mathcal{U}_y = -e^{-\varepsilon^p y}$$

Method of dominant balance yields:

$$-2p+1 = -p, \quad \boxed{p=1}; \qquad \boxed{x = \varepsilon y}$$

$$\begin{cases} \mathcal{U}_{yy} + \mathcal{U}_y = -\varepsilon e^{-\varepsilon y} \sim -\varepsilon \\ \mathcal{U}(0) = 0 \end{cases}$$

Solving this equation, we get:

$$\mathcal{U} \sim -\varepsilon y + C_1 e^{-y} + C_2 \Rightarrow C_1 = -C_2$$

$$\Rightarrow \mathcal{U} \sim -\varepsilon y + C_1(e^{-y} - 1),$$ where C_1 is to be determined.

Outside the boundary layer, the solution is changing slowly so we may estimate:

$$u \sim e^{-x} - e^{-1} + 1 \text{ (as before)}$$

It remains to determine C_1. To do so, we apply the <u>Matching principle</u>:

$$\lim(\text{outer sol'n as } x \to 0) = \lim(\text{inner as } y \to \infty)$$

Outer: $u \sim e^{-x} - e^{-1} + 1 \sim 2 - x - e^{-1}, \quad x \to 0$

Inner:

$$U \sim -\varepsilon y + C_1 (e^{-y} - 1)$$

$$\sim -\varepsilon y - C_1, \quad y \to \infty$$

$$\sim -x - C_1$$

Matching:

$$2 - x - e^{-1} \sim -x - C_1$$

$$\Rightarrow \boxed{C_1 = -2 + e^{-1}}$$

So we have:

$$u \sim \begin{cases} e^{-x} - e^{-1}, & x \gg \varepsilon \quad \text{[outer sol'n]} \\ -x + (-2 + e^{-1})(e^{-\frac{x}{\varepsilon}} - 1), & x \ll \varepsilon \quad \text{[inner sol'n]} \end{cases}$$

Composite solution: A uniform solution can be obtained from u_{outer} and U_{inner} as:

$$u_{unif} = u_{outer} + u_{inner} - u_{common}:$$

Here, $u_{common} \sim 1 - x - e^{-1}$ so that

$$u_{unif} = \underbrace{e^{-x} - e^{-1} + 1}_{\text{outer}} + \underbrace{(-2 + e^{-1})(e^{-\frac{x}{\varepsilon}} - 1) - x}_{\text{inner}} - \underbrace{(-x + 2 - e^{-1})}_{\text{common}}$$

$$\sim e^{-x} - e^{-1} + 1 + (-2 + e^{-1}) e^{-\frac{x}{\varepsilon}}$$

Remark: The composite sol'n is often better than either u_{outer} or U_{inner}

Higher-order matching:

Outer

Expand $u = u_0 + \varepsilon u_1 + \cdots$;

$$\begin{cases} u_0' = -e^{-x}, & u_0(1) = 1 \\ u_0'' + u_1' = 0, & u_1(0) = 0 \end{cases}$$

$$\Rightarrow \quad u_0 = e^{-x} - e^{-1} + 1$$

$$u_1 = +e^{-x} + e^{-1}$$

Inner: Expand $u = \mathcal{U}_0(y) + \varepsilon \mathcal{U}_1(y) + \cdots$, $y = \frac{x}{\varepsilon}$

$$\mathcal{U}_{yy} + \mathcal{U}_y = -\varepsilon e^{-\varepsilon y} = -\varepsilon(1 - \varepsilon y \ldots)$$

$$\Rightarrow \begin{cases} \mathcal{U}_{0yy} + \mathcal{U}_{0y} = -\varepsilon & \mathcal{U}_0(0) = 0 \\ \mathcal{U}_{1yy} + \mathcal{U}_{1y} = \varepsilon y & \mathcal{U}_1(0) = 0 \end{cases}$$

$$\Rightarrow \mathcal{U}_0 = -\varepsilon y + C_1(e^{-y} - 1)$$

$$\mathcal{U}_1 = \frac{\varepsilon}{2} y^2 - \varepsilon y + C_2(e^{-y} - 1)$$

Matching: $u_{outer} = e^{-x} - e^{-1} + 1 + \varepsilon(e^{-x} + e^{-1})$

$$= 2 - x - e^{-1} + \varepsilon(-1 + e^{-1} + x), \quad x \to 0$$

$$\mathcal{U}_{inner} = -\varepsilon y + C_1(e^{-y} - 1) + \frac{\varepsilon^2 y^2}{2} - \varepsilon^2 y + \varepsilon C_2(e^{-y} - 1)$$

$$\sim -x - C_1 + \frac{x^2}{2}$$

Matching:

$$u_{inner} = -\varepsilon y + C_1(e^{-y}-1) + \frac{\varepsilon^2 y^2}{2} - \varepsilon^2 y + \varepsilon C_2(e^{-y}-1)$$

$$\sim -x - C_1 + \frac{x^2}{2} - \varepsilon x - \varepsilon C_2 \,, \qquad \begin{array}{l} y \to \infty \\ x = \varepsilon y \end{array}$$

$$\sim -x + \frac{x^2}{2} - C_1 + \varepsilon(-x - C_2)$$

$$u_{outer} = e^{-x} - e^{-1} + 1 + \varepsilon(+e^{-x} - e^{-1})$$

$$\sim 1 - x + \frac{x^2}{2} - e^{-1} + 1 + \varepsilon(+1 - x - e^{-1}) \,, \quad x \to 0$$

$$\sim -x + \frac{x^2}{2} + 2 - e^{-1} + \varepsilon(-x + 1 - e^{-1})$$

$$\Rightarrow \begin{cases} -C_1 = 2 - e^{-1} \\ -C_2 = 1 - e^{-1} \end{cases} \Rightarrow \begin{array}{l} C_1 = -2 + e^{-1} \\ C_2 = -1 + e^{-1} \end{array}$$

Composite sol'n:

$$u_{composite} = u_{outer} + u_{inner} - u_{common}$$

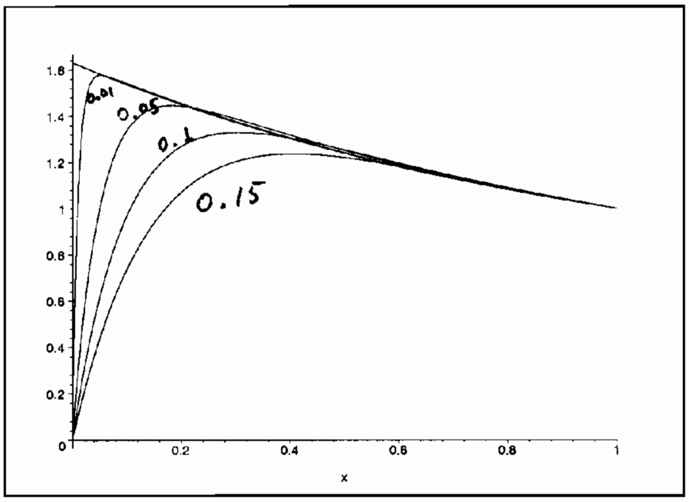
$$= e^{-x} - e^{-1} + 1 + \varepsilon(e^{-x} - e^{-1})$$

$$+ \left(-x - C_1 + C_1 e^{-\frac{x}{\varepsilon}}\right) + \frac{x^2}{2} - \varepsilon x + \varepsilon\left(-C_2 + C_2 e^{-\frac{x}{\varepsilon}}\right)$$

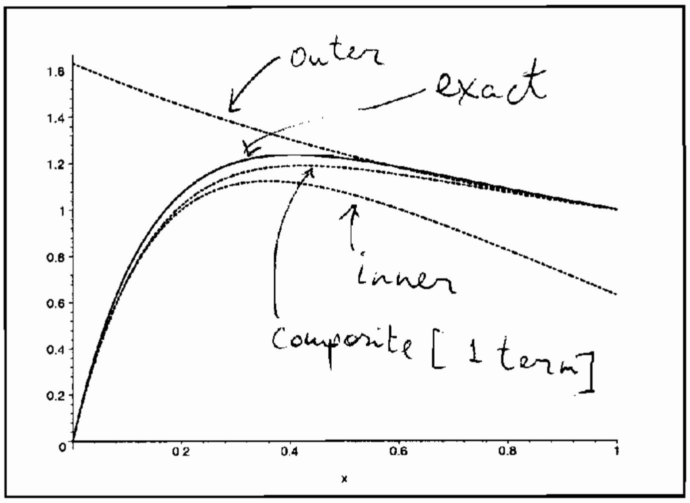
$$- \left(-x + \frac{x^2}{2} - C_1 + \varepsilon(-x - C_2)\right)$$

$$u = e^{-x} - e^{-1} + 1 + (-2 + e^{-1}) e^{-\frac{x}{\varepsilon}} + \varepsilon\left(e^{-x} - e^{-1} + (-1 + e^{-1}) e^{-\frac{x}{\varepsilon}}\right)$$

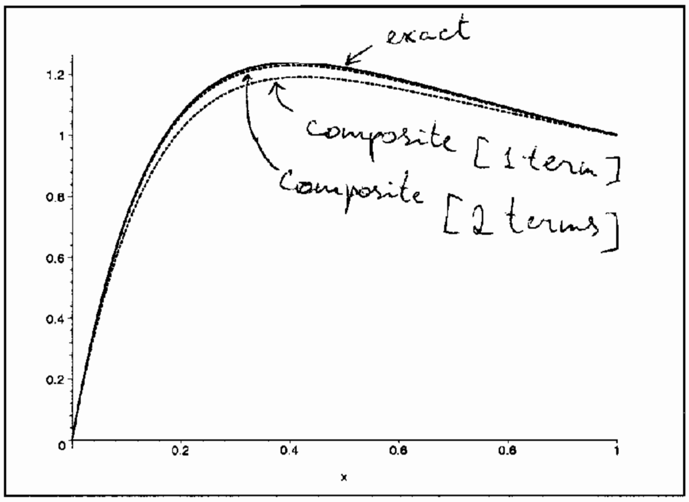
Exact sol'n with ε as indicated



$\varepsilon = 0.15$



$\varepsilon = 0.15$



Where is the boundary layer?

- The answer is determined through consistency of matching.
- Internal layers may exist; answer ~~ eq~~ may not be unique, especially for <u>non-linear</u> probs.

<u>Example</u>: <u>Consider</u>:

$$\begin{cases} \varepsilon u'' + a(x) u' = b'(x)\, a(x) \\ u(0) = A, \quad u(1) = B \\ a(x) > 0 \end{cases}$$

First, let's try to construct <u>outer sol'n</u>:

$$a u' \sim b, \qquad u_0 = \qquad b(x) + C$$

If we assume a BL at <u>$x=0$</u>, then $u_{outer}(1) = B$

$$\Rightarrow C = B - b(1).$$

$$\Rightarrow u_0 = b(x) - b(1) + B.$$

<u>Inner sol'n</u>: $y = \frac{x}{\varepsilon}$, $u(x) = U(y)$;

$$U_{yy} + U_y\, a(\varepsilon y) = \varepsilon b'(\varepsilon y)\, a(\varepsilon y)$$

$$\Rightarrow U_{yy} + a(0) U_y \sim \varepsilon b'(0)\, a(0)$$

$$\Rightarrow U \sim C_1 e^{-a(0) y} + y\, b(0) \qquad + C_2$$

$$\Rightarrow U \sim C_1 \left(e^{-a(0) y} - 1 \right) + A + \varepsilon y\, b'(0)$$

for C_1 to be determined.

Matching: Since $a > 0$, $e^{-a(0)y} \to 0$ as $y \to \infty$

so $\mathcal{U} \sim -C_1 + A + \underbrace{\cancel{\varepsilon} x\, b'(0)}$, $y \to \infty$

$u \sim b(0) + \underbrace{\cancel{\varepsilon} x\, b'(0)} - b(1) + B$

$\Rightarrow -C_1 + A = b(0) - b(1) + B$

Matching is consistent;

Uniform sol'n: $u \sim b(x) - b(1) + B + \left(A - b(0) + b(1) - B\right) e^{-a(0)\frac{x}{\varepsilon}}$

On the other hand, if we assume a boundary layer at $x = 1$, then the inner sol'n becomes:

$u(x) = \mathcal{U}(y)$ where $\boxed{y = \frac{1-x}{\varepsilon}}$, $y > 0$

$\Rightarrow \mathcal{U}_{yy} - a(1)\,\mathcal{U}_y \sim \varepsilon\, b'(1)\, a(1)$

$\Rightarrow \mathcal{U}_y \sim C_1 \Big(\underbrace{e^{+a(1)y}}_{\text{blows up as } y \to \infty} - 1 \Big) + B + \varepsilon y\, b'(1)$

$\to \mathcal{U}$ blows up in the limit $y \gg 1$ because of $e^{+a(1)y}$

$\Rightarrow$ No consistent matching possible!!

Conclusions: For pblms of the form:

$$\varepsilon u'' + a(x) u' + b(x) u = c(x), \quad x \in [0,1]$$

with $a(x) \neq 0$, BL occurs at $x = 0$ if $a > 0$
(of $O(\varepsilon)$) at $x = 1$ if $a < 0$

- What if $a = 0$ at 0 or 1? [Leads to different BL scaling]
- What if $a = 0$ inside $[0,1]$?

["interior" BL]

Example:

$$\begin{cases} \varepsilon u'' + x u' - x u = 0 \\ u(0) = 0, \quad u(1) = e \end{cases}$$

- No BL at $x = 1$ [Why?]
- Outer sol'n: $u' - u \sim 0 \Rightarrow \boxed{u \sim e^x}$

Inner scaling: $x = \varepsilon^p y$, $u = \mathcal{U}(y)$

$$\varepsilon^{1-2p} \mathcal{U}_{yy} + y \mathcal{U}_y - \varepsilon^p y \mathcal{U} = 0$$

Dominant balance: $\boxed{p = \frac{1}{2}}$

$$\Rightarrow \mathcal{U}_{yy} + y \mathcal{U}_y \sim 0 \Rightarrow \mathcal{U} =$$

<u>Example</u>: $\varepsilon u'' + x u' - x u = 0$
$u(0) = 0, \quad u(1) = e$

- No BL at $x = 1$ [Why?]
- <u>Outer sol'n</u>: $u' \sim u \Rightarrow u = e^x$

<u>Inner scaling</u>: $x = \varepsilon^p y \qquad u(x) = \mathcal{U}(y)$

$$\varepsilon^{1-2p}\, \mathcal{U}_{yy} + y\, \mathcal{U}_y - \varepsilon^p y\, \mathcal{U} = 0$$
$$\text{①} \qquad \text{②} \qquad \text{③}$$

Balancing ① & ② we get $\boxed{p = \frac{1}{2}}$;

to leading order, $\mathcal{U}_{yy} + y\, \mathcal{U}_y \sim 0$

$$\Rightarrow \quad \mathcal{U}(y) \sim A_0 \int_0^y e^{-t^2/2}\, dt \, .$$

<u>Matching</u>:

$$u \sim e^x = e^{\varepsilon^{\frac{1}{2}} y} \sim 1 + \varepsilon^{\frac{1}{2}} y + \cdots, \quad x \to 0$$

$$\mathcal{U}(y) \sim A_0 \sqrt{\frac{\pi}{2}}, \quad y \to \infty$$

$$\Rightarrow \boxed{A_0 = \sqrt{\frac{2}{\pi}}}$$

<u>Composite</u>:
$$u \sim e^x + \sqrt{\frac{2}{\pi}} \int_0^{x/\varepsilon^{\frac{1}{2}}} e^{-t^2/2}\, dt - 1$$

Ex: $\varepsilon u'' - (1-x^2)u = -1$, $u(\pm 1) = 0$, $x \in [-1,1]$

Inner: $u \sim \dfrac{1}{1-x^2}$; Note that $u \to \infty$ as $x \to \pm 1$!

Outer: Near $x = 1$: rescale $x = 1 - \varepsilon^p y$

$u(x) = \varepsilon^q \mathcal{U}(y)$

then $1 - x^2 = 2\varepsilon^p y + \ldots$

$$\Rightarrow \varepsilon^{1-2p+q}\,\mathcal{U}_{yy} - 2\varepsilon^{p+q} y\,\mathcal{U} \sim -1$$

Balance: $1 - 2p = p \Rightarrow \boxed{p = \frac{1}{3}}$

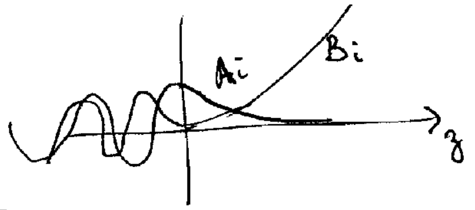
and $p + q = 1 \Rightarrow \boxed{q = -\frac{1}{3}}$

$$\Rightarrow \mathcal{U}_{yy} - 2y\,\mathcal{U} \sim -1 \qquad (*)$$

Sol'n is given by

$$\mathcal{U} = \mathcal{U}_p + C_1 A_i(2^{\frac{1}{3}} y) + C_2 B_i(2^{\frac{1}{3}} y)$$

where A_i, B_i ~~is A~~ solves Airy ODE: $\left\{ f''(z) - z f(z) = 0 \right.$



$A_i(z) \to 0$ as $z \to \infty$

$B_i(z) \to \infty$ as $z \to \infty$ $\Rightarrow \boxed{C_2 = 0}$

and $\mathcal{U}_p$ is a particular sol'n of $(*)$

Matching: **Outer:** $u \sim \dfrac{1}{(1-x)(1+x)}$, $x \to 1$

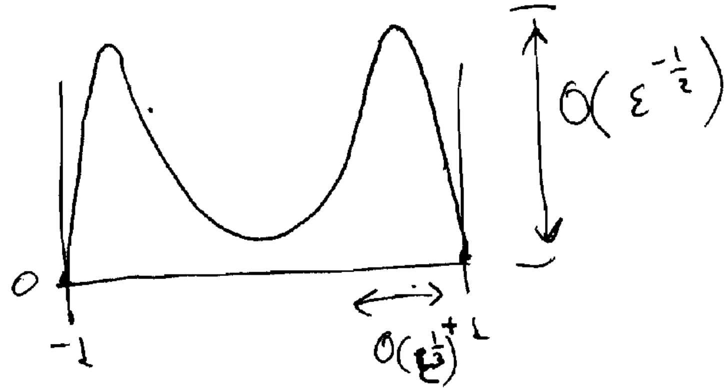
$$\sim \frac{1}{2(\varepsilon^{\frac{1}{3}} y)} + \cdots$$

Inner: $u = \varepsilon^{-\frac{1}{3}} U(y)$

$$\sim \frac{1}{\varepsilon^{\frac{1}{3}}} \underbrace{U_p(y)}_{\sim \frac{1}{2y}}, \quad y \to \infty$$

C_1 is given by: $U_p(0) + C_1 A_i(0) = 0$

By symmetry, the same BL at $x = -1$:



Internal BL: $\begin{cases} \varepsilon u'' + \underbrace{\left(x-\frac{1}{2}\right)}_{a(x)} u' - \left(x-\frac{1}{2}\right) u = 0 \\ u(0) = \alpha \quad u(1) = e \end{cases}$

Note: $a(x) < 0$ at 0 and $a(x) > 0$ at 1

$\Rightarrow$ No BL possible at $x = 0, 1$.

Outer sol'n: $u' = u \Rightarrow u = e^x$ satisfies $u(1) = e$

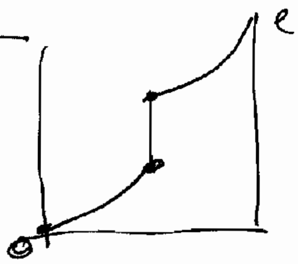
or $u = \alpha e^x$ satisfies $u(0) = 0$

but impossible to satisfy both!

$\Rightarrow \exists$ exists internal BL

at $x = \frac{1}{2}$

(where $a(x) = 0$)



Near $x = \frac{1}{2}$ rescale: $x = \frac{1}{2} + \varepsilon^p y$, $u(x) = \mathcal{U}(y)$

$$\Rightarrow \varepsilon^{1-2p} \mathcal{U}_{yy} + y\mathcal{U}_y - \varepsilon^p y \mathcal{U} = 0$$

$\Rightarrow$ as before, $p = \frac{1}{2}$, $\mathcal{U}_{yy} + y\mathcal{U}_y = 0$

$$\Rightarrow \mathcal{U} \sim C_1 + C_2 \int_0^y e^{-s^2/2}\, ds$$

Matching:

$\underline{x < \frac{1}{2}}$: $\quad u \sim \alpha e^{x} \to \alpha e^{\frac{1}{2}}, \quad x \to \frac{1}{2}^{-}$

$\underline{x > \frac{1}{2}}$: $\quad u \sim e^{x} \to e^{\frac{1}{2}}, \quad x \to \frac{1}{2}^{+}$

Inner:

$$u \sim c_1 + c_2 \int_0^y e^{-\frac{s^2}{2}} \to \begin{cases} c_1 + \sqrt{\frac{\pi}{2}}\, c_2, & y \to \infty \\ c_1 - \sqrt{\frac{\pi}{2}}\, c_2, & y \to -\infty \end{cases}$$

$$\Rightarrow \quad \begin{cases} c_1 + \sqrt{\frac{\pi}{2}}\, c_2 = e^{\frac{1}{2}} \\ c_1 - \sqrt{\frac{\pi}{2}}\, c_2 = \alpha e^{\frac{1}{2}} \end{cases} \Rightarrow$$

$$\Rightarrow \quad c_1 = e^{\frac{1}{2}} \left(\frac{1+\alpha}{2} \right)$$

$$c_2 = e^{\frac{1}{2}} \left(\frac{1-\alpha}{2} \right) \sqrt{\frac{2}{\pi}}$$

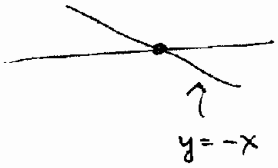
Composite:

$$\boxed{u_{comp} = e^{x} \left(\frac{1+\alpha}{2} + \frac{1-\alpha}{2} \sqrt{\frac{2}{\pi}} \int_0^{(x-\frac{1}{2})/\varepsilon} e^{-s^2/2}\, ds \right)}$$

Exponential ill-conditioning:

$$\begin{cases} \varepsilon u'' - x u' = 0 \\ u(-1) = 0, \quad u(b) = 1, \quad b > 0 \end{cases}$$

- Expect BL at endpts [not interior]



Outer: $x u' \sim 0 \Rightarrow u = \alpha$

Inner, $x = -1$: $x = -1 + \varepsilon y \Rightarrow u(x) = U_e(y)$

$$U_{e\,yy} + U_{e\,y} = 0 \Rightarrow U_e \sim \alpha(1 - e^{-y})$$

Inner, $x = +1$: $x = b - \varepsilon y \Rightarrow U_R \sim \alpha + [1-\alpha] e^{-yb}$

Composite: $u_c \sim \alpha - \alpha e^{-\varepsilon^{-1}(x+1)} + (1-\alpha) e^{-\varepsilon^{-1} b (b-x)}$

Problem: α is undetermined!!

- To determine α, we need to consider exponentially small, $O(e^{-\frac{1}{\varepsilon}})$ terms.
- This makes the problem numerically difficult since $e^{-\frac{1}{\varepsilon}} <$ machine precision even if $\varepsilon = 0.02$ $\left(e^{-\frac{1}{0.02}} = 2 \times 10^{-20}\right)$

To determine α analytically, we apply a <u>solvability condition</u>: For a test fcn $u^*(x)$ we have:

$$\int (\varepsilon u'' - x u') u^* = 0 \quad ; \quad \text{integrating by parts:}$$

$$\int u\left(\varepsilon u^{*''} + (x u^*)'\right) + \varepsilon\left(u^* u' - u u^{*'}\right)\Big|_{-1}^{b} - x u u^*\Big|_{-1}^{b} = 0$$

Now <u>choose</u> u^* to be a solution of

$$\varepsilon u^{*''} + (x u^*)' = 0$$

Particular sol'n is found to be

$$\boxed{u^* = e^{-\frac{x^2}{2\varepsilon}}}$$

By estimating $u \sim u_c$, we find:

$\varepsilon u'(-1) = \alpha$, $u(-1) = 0$

$\varepsilon u'(b) = b(1-\alpha)$ $u(b) = 1$

$u^*(-1) = e^{-\frac{1}{2\varepsilon}}$ $u^{*'}(-1) = \frac{1}{\varepsilon} e^{-\frac{1}{2\varepsilon}}$

$u^*(b) = e^{-\frac{b^2}{2\varepsilon}}$ $u^{*'}(b) = -\frac{b}{\varepsilon} e^{-\frac{b^2}{2\varepsilon}}$

So that $\varepsilon(u^* u' - u u^{*'})\Big|_{-1}^{b}$

$$= e^{-\frac{b^2}{2\varepsilon}}\left(b(1-\alpha) + b\right) - e^{-\frac{1}{2\varepsilon}}\alpha \quad ;$$

$$x u u^*\Big|_{-1}^{b} = b\, e^{-\frac{b^2}{2\varepsilon}} \quad ;$$

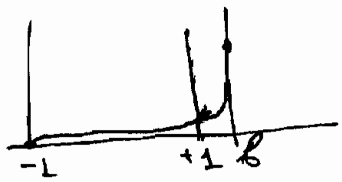
and $\varepsilon(u^* u' - u u^{*'}) - x u u^*\Big|_{-1}^{b} = 0$

$$\Rightarrow \quad e^{-\frac{b^2}{2\varepsilon}}(2b - \alpha b) - e^{-\frac{1}{2\varepsilon}}\alpha - b\, e^{-\frac{b^2}{2\varepsilon}} = 0$$

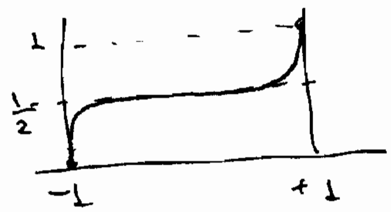
$$\Rightarrow \quad \alpha = \frac{e^{-\frac{b^2}{2\varepsilon}}\, b}{b\, e^{-\frac{b^2}{2\varepsilon}} + e^{-\frac{1}{2\varepsilon}}}$$

$$\boxed{\alpha = \frac{1}{1 + \frac{1}{b}\exp\left(\frac{b^2 - 1}{2\varepsilon}\right)}}$$

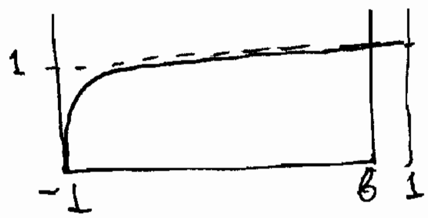
Thus, • $\alpha \sim 0$ if $b > 1$:



• $\alpha \sim \frac{1}{2}$ if $b \sim 1$:



• $\alpha \sim 1$ if $b < 1$:



• "Sudden" jump as b crosses 1 !