Delay Differential Equations

Consider a DDE $\quad y' = f(t, y, y(t-\tau))$

The delay τ can cause instability, even if the ODE is stable when $\tau = 0$. Typically, increasing τ will result in Hopf bifurcation, possibly followed by period doubling and/or chaos.

<u>Ex</u>: Consider $\quad y'(t) = -y(t-\tau)$

- Stable if $\tau = 0$ with $y \to 0$
- For $\tau > 0$, sub in $y = e^{\lambda t}$

$$\Rightarrow \quad \boxed{\lambda = -e^{-\lambda \tau}}$$

Looking for Hopf bifurcation, we substitute

$$\lambda = i\omega \quad \Rightarrow \quad i\omega = -\cos\omega\tau + i\sin\omega\tau$$

$$\Rightarrow \begin{cases} \cos\omega\tau = 0 \\ \omega = \sin\omega\tau \end{cases}$$

$$\Rightarrow \omega\tau = \frac{\pi}{2} + 2\pi n \Rightarrow \omega = 1 \Rightarrow \tau = \frac{\pi}{2} + 2\pi n$$

<u>or</u> $\quad \omega\tau = -\frac{\pi}{2} + 2\pi n \Rightarrow \omega = -1, \quad \tau = -\frac{\pi}{2} + 2\pi n$

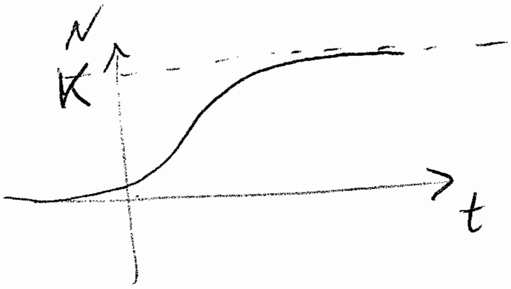
So $\exists$ infinitely many Hopf bifurcations, the first one occurs at $\boxed{\tau = \frac{\pi}{2}}$

Logistic equation:

$$\frac{dN}{dt} = r N\left(1 - \frac{N}{K}\right)$$

- Describes population growth with growth rate r and carrying capacity K.
- $N = K$ is stable s.s.;

  $N = 0$ is unstable s.s.



Delay:

$$\frac{dN}{dt} = r N\left(1 - \frac{N(t-\tau)}{K}\right)$$

- τ accounts for hatching / food absorbtion / maturation lag.

Non-dimensional Logistic d.d.e.:

$$\frac{dy(t)}{dt} = \lambda y(t)\left(1 - y(t-1)\right), \quad \lambda = r\tau$$

S.S.: $y_0 = 0$ and $y_0 = 1$.

Stability of $y_0 = 1$: $\quad y = 1 + \varepsilon e^{\sigma t}$

$$\Rightarrow \quad \sigma = -\lambda e^{-\sigma}$$

Set $\sigma = i\omega$; then $i\omega = -\lambda(\cos\omega - i\sin\omega)$

$$\boxed{\omega = \frac{\pi}{2}, \quad \lambda = \frac{\pi}{2}}$$

Near Hopf: Expand: $y = 1 + \varepsilon y_1(s) + \varepsilon^2 y_2(s) + \cdots$

Let $s = \omega t$, $y(t) = Y(s)$ $\quad \lambda = \lambda_0 + \varepsilon^2 \lambda_1 + \cdots$

$\Rightarrow y(t-1) = Y(s-\omega)$

$$\omega = \omega_0 + \varepsilon^2 \omega_1 + \cdots$$

$$s = \omega t$$

where $\lambda_0 = \frac{\pi}{2}$, $\omega_0 = \frac{\pi}{2}$

Now $\lambda y(1 - y(t-1)) = \lambda Y(1 - Y(s-\omega))$

$$= (\lambda_0 + \varepsilon^2 \lambda_1)\cdot(1 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \cdots)\cdot(-\varepsilon y_1(s-\omega) - \varepsilon^2 y_2(s-\omega)$$

$$= \varepsilon(-\lambda_0 y_1(s-\omega))$$

$$+ \varepsilon^2(-\lambda_0 y_2(s-\omega) - \lambda_0 y_1 y_1(s-\omega))$$

$$+ \varepsilon^3(-\lambda_0 y_1 y_2(s-\omega) - \lambda_0 y_2 y_1(s-\omega) - \lambda_1 y_1(s-\omega) - \lambda_0 y_3(s-\omega)$$

Moreover, $y_1(s-\omega) = y_1(s - \omega_0 - \varepsilon^2 \omega_1 + \cdots)$

$$= y_1(s-\omega_0) - \varepsilon^2 \omega_1 y_1'(s-\omega_0)$$

which contributes an extra terms at $O(\varepsilon^3)$

$$\therefore = \varepsilon(-\lambda_0 y_1(s-\omega_0)) + \varepsilon^2(-\lambda_0 y_2(s-\omega_0) - \lambda_0 y_1 y_1(s-\omega_0))$$

$$+ \varepsilon^3\Big(-\lambda_0 y_1 y_2(s-\omega_0) - \lambda_0 y_2 y_1(s-\omega_0) - \lambda_1 y_1(s-\omega_0)$$

$$-\lambda_0 y_3(s-\omega_0) \qquad + \lambda_0 \omega_1 y_1'(s-\omega_0)\Big)$$

Now $\frac{d}{dt} = \omega \frac{d}{ds}$

$$\Rightarrow \frac{d}{dt} y = (\omega_0 + \varepsilon^2 \omega_1)(\varepsilon y_1' + \varepsilon^2 y_2')$$

$$= \varepsilon(\omega_0 y_1') + \varepsilon^2(\omega_0 y_2') + \varepsilon^3(\omega_0 y_3' + \omega_1 y_1')$$

So we get:

$O(\varepsilon)$: $\omega_0 y_1' = -\lambda_0 y_1(s-\omega_0)$

$O(\varepsilon^2)$: $\omega_0 y_2' = -\lambda_0 y_2(s-\omega_0) - \lambda_0 y_1 y_1(s-\omega_0)$

$O(\varepsilon^3)$: $\omega_0 y_3' = -\lambda_0 y_3(s-\omega_0) - \lambda_0 y_1 y_2(s-\omega_0) - \lambda_0 y_2 y_1(s-\omega_0)$

$- \lambda_1 y_1(s-\omega_0) + \lambda_0 \omega_1 y_1'(s-\omega_0)$

$- \omega_1 y_1'(s)$

$O(\varepsilon)$: $y_1 = e^{is} + c.c.;\ \omega_0 = \frac{\pi}{2},\ \lambda_0 = \frac{\pi}{2}$

$O(\varepsilon^2)$: $y_1(s-\omega_0) = e^{i(s-\frac{\pi}{2})} = -ie^{is}$

$\Rightarrow y_2' + y_2 = ie^{2si} \Rightarrow y_2 = pe^{2si}$, $2pi + p = i$, $p = \frac{i}{2i-1}$

$y_2 = \frac{i}{2i-1} e^{2si} + c.c.$

$O(\varepsilon^3)$:

$$y_1 \, y_2(s-\omega_0) =$$

$$= \left(e^{is} + e^{-is}\right)\left(\frac{-i}{2i-1} e^{+2s} + c.c.\right)$$

$$= \frac{-i}{2i-1} e^{is} + c.c. + \dots ;$$

Solvability: $\quad y_1(s-\omega_0)\, y_2(s) = \frac{1}{2i-1} e^{is} + \dots$

$$\lambda_0 \underbrace{\left(\frac{-i}{2i-1} - \frac{1}{2i-1}\right)}_{\underbrace{\frac{(-i-1)(2i+1)}{5}}_{\frac{+1-3i}{5}}} + \lambda_1 i + \lambda_0 \omega_1 - i\omega_1 = 0$$

$$\Rightarrow \boxed{\omega_1 = -\frac{1}{5}, \quad \lambda_1 = \frac{3\pi-2}{10}}$$

So near hopf bifurcation, we get:

$$y \sim 1 + \varepsilon \cos(\omega t), \quad \omega = \frac{\pi}{2} - \varepsilon^2 \frac{1}{5} + \dots$$

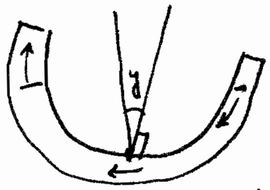
$$\lambda \sim \frac{\pi}{2} + \varepsilon^2 \frac{3\pi-2}{10}$$

Or: $\quad \varepsilon = \sqrt{\left(\lambda - \frac{\pi}{2}\right) \frac{10}{3\pi-2}} \; ;$

$$\begin{cases} y \sim 1 + \sqrt{\left(\lambda - \frac{\pi}{2}\right)\frac{10}{3\pi - 2}} \cos(\omega t) \\ \omega \sim \frac{\pi}{2} - \left(\lambda - \frac{\pi}{2}\right)\frac{2}{3\pi - 2} \end{cases}$$

Mechanical Vibrations

Minorski's Model of ship rolling and control [1947]:



Ship hull

- Ballast tanks on sides of ship hull are filled with water
- Water is pumped from one side to other to counteract rolling.

$$y'' + \varepsilon y' + y = -\varepsilon b_1 y'(t-\tau) + \varepsilon c \, y'^3(t-\tau)$$

- τ models the delay in reaction time.
- if $b_1 = 0$, $c=0$ then $y \sim e^{-\frac{\varepsilon}{2}t} \sin\left(t + O(\varepsilon^2)\right)$
- The hope is that increasing b_1 will ~~stab~~ increase stability [if $\tau = 0$, $c = 0$ $\Rightarrow$ $y \sim e^{-\frac{\varepsilon + b_1}{2}t} \sin(t)$] but the lag τ can cause problems.

Multiple scales:

$$y = Y(t, s), \quad s = \varepsilon t$$

$$y(t-\tau) = Y(t-\tau,\ s-\varepsilon\tau)$$

$$= Y(t-\tau, s) - \varepsilon\tau Y_s(t-\tau, s) + \dots$$

$$Y = Y_0 + \varepsilon Y_1$$

$$Y_{0tt} + Y_0 = 0 \Rightarrow Y_0 = A(s)\, e^{it} + c.c.$$

$$Y_{1tt} + Y_{1t} = -2Y_{0st} - Y_{0t} - b_1 Y_{0t}(t-\tau) + c\, Y_{0t}^3(t-\tau)$$

$$= \left(-2iA' - iA - ib_1 A e^{-i\tau}\right) e^{it} + c\left(Ai\, e^{it-i\tau} - i\bar{A}\, e^{-it+i\tau}\right)^3$$

$$= e^{it}\left(-2iA' - iA - ib_1 A\, e^{-i\tau} + 3cA^2 \bar{A}\, i\, e^{-i\tau}\right) + c.c.$$

$$\Rightarrow \boxed{2A_s = -A - b_1 A e^{-i\tau} + 3c A^2 \bar{A} e^{-i\tau}}$$

Introduce $A(s) = R(s) e^{i\varphi(s)}$

$$\Rightarrow \begin{cases} 2R_s = -(1 + b_1 \cos\tau) R + 3cR^3 \cos\tau \\ 2\varphi_s = b_1 \sin\tau - 3cR^2 \sin\tau \end{cases}$$

Note: $R = 0$ is stable whenever

$1 + b_1 \cos\tau > 0$;

unstable otherwise.

So $\boxed{b_{1H} = \frac{-1}{\cos\tau}}$; Need $\cos\tau < 0$

$\Rightarrow \tau > \frac{\pi}{2}$

As τ crosses b_{1H}, a pitchfork bifurcation occurs since $3c > R$

Non-steady state $R = \sqrt{\frac{1 + b_1 \cos\tau}{3c\cos\tau}}$

$$\Rightarrow \quad 2\frac{dA}{ds} = -A - A(s-\theta)\exp(-i\tau)\left[b_1 - 3c|A(s-\theta)|^2\right]$$

Setting $A = Re^{i\varphi}$; <u>assume</u> $\tau = (1+2n)\pi$ [for simplicity]

$$\Rightarrow \begin{cases} 2R_s = -R + R(s-\theta)\left[b_1 - 3cR^2(s-\theta)\right]\cos(\varphi(s-\theta)-\varphi(s)) \\ 2R\varphi_s = R(s-\theta)\left[b_1 - 3cR^2(s-\theta)\right]\sin(\varphi(s-\theta)-\varphi(s)) \end{cases}$$

Now $b_{1h} = \frac{-1}{\cos(\tau)} = 1$;

as $b_1 > b_{1h}$, $R=0$ bifurcates into

$$R_e = \sqrt{\frac{b_1 - 1}{3c}} \quad ; \quad \varphi_e \equiv \text{const}.$$

<u>Linear stability of R_e</u>:

Let $R = R_e + c_1 e^{\lambda t}$

$\varphi = \varphi_e + c_2 e^{\lambda t}$

$$\Rightarrow \begin{cases} 2\lambda = -1 + e^{-\lambda\theta}(-2b_1 + 3) & (*) \\ 2\lambda = \exp(-\lambda\theta) - 1 & (**) \end{cases}$$

If λ is real, then only $\lambda = 0$ is possible as sol'n to $(*)$

Hopf: $\lambda = i\sigma$

$\Rightarrow$ (*) becomes:

$$0 = -1 + \cos(\sigma\theta)(-2b_1+3)$$
$$2\sigma = -\sin(\sigma\theta)\,(-2b_1+3)$$

Eliminate b_1: $\begin{cases} \tan\sigma\theta = -2\sigma \\ 2b_1 = 3 - \frac{1}{\cos(\sigma\theta)} \end{cases}$

Large θ: then $\sigma\theta \sim \pi$, $b_1 \sim 2$

[Secondary bif]

Period-doubling:

If $\varphi \sim$ const we get

$$2R' = -R + R(s-\theta)\left[b_1 - 3cR^2(s-\theta)\right]$$

When $\theta \gg 1$, we can rescale $s = S\theta$,

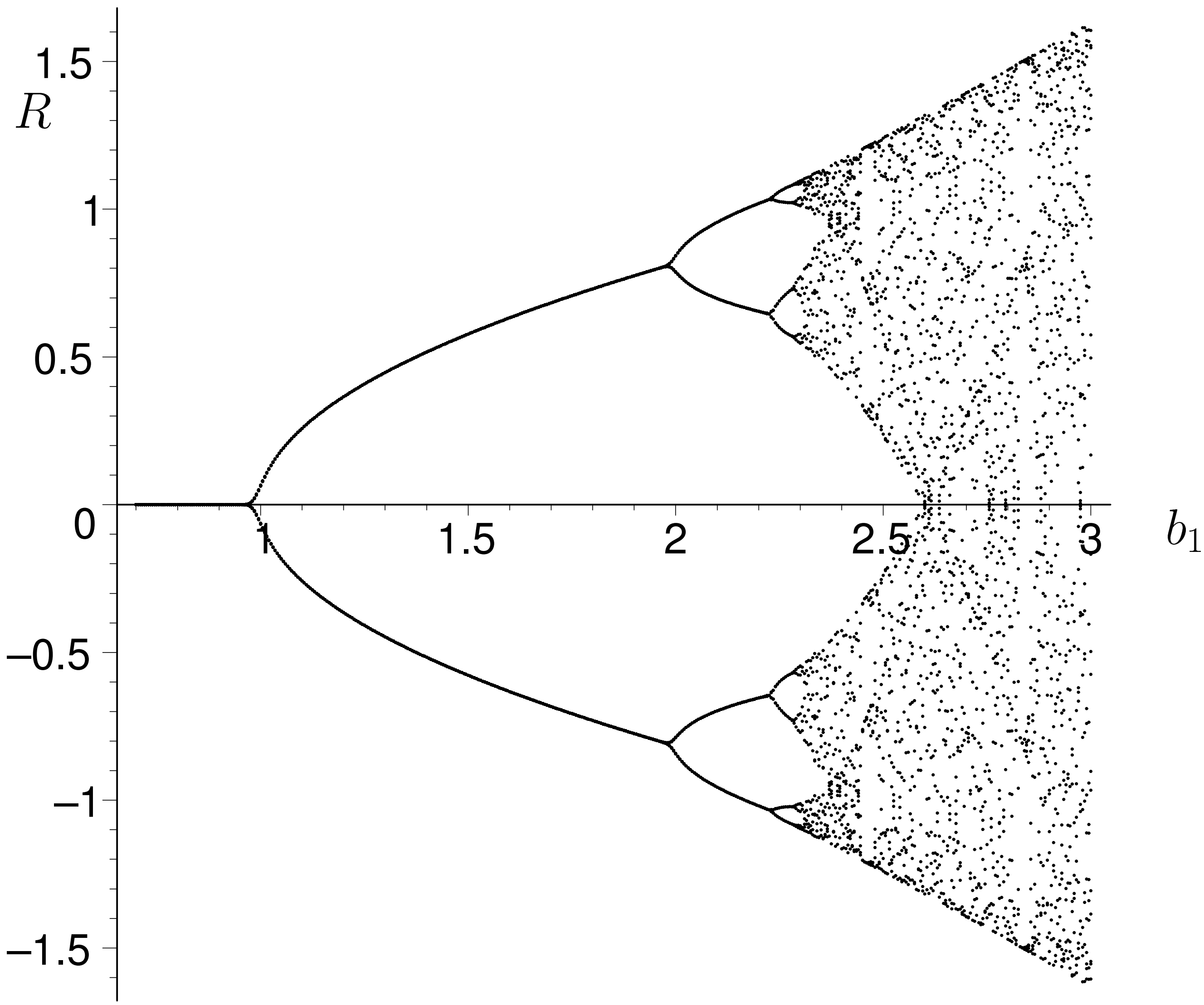
$R(s) = \hat{R}(S)$

$\Rightarrow R' = \frac{1}{\theta}\hat{R}'(S) \ll 1$

So we get:

$$\boxed{\hat{R} = \hat{R}(S-1)\left[b_1 - 3c\hat{R}^2(S-1)\right]}$$

This discrete map has a period-doubling route to chaos:



Period doubling and chaos for large θ. Note that the first two bifurcations occur at $b_1 = 1$ and $b_1 = 2$. Here, $c = 0.5$.