MATH 5250/4250, Homework2

Due: Tuesday, 30 Sep.

1. The equation for the transverse displacement $u = u(x)$ of a nonlinear beam subject to an axial load λ is

$$u_{xxxx} + \left(\lambda - \frac{1}{4}\int_0^1 u_x^2 dx\right) u_{xx} = 0, \quad 0 < x < 1$$

with boundary conditions $u = u_{xx} = 0$ at $x = 0, 1$.

a) At what value of $\lambda = \lambda_c$ does the constant steady state $u = 0$ bifurcate to a non-constant solution?

b) Derive the bifurcation diagram near $\lambda = \lambda_c$.

2. a) Let $r = \sqrt{x^2 + y^2}$ and suppose that $u(x, y) = U(r)$ for some function U. Show that

$$\Delta u = U_{rr} + \frac{1}{r} u_r.$$

b) Let $x = r\cos\theta$; $y = r\sin\theta$ and suppose that $u(x, y) = U(r, \theta)$. Show that

$$\begin{aligned} u_x &= U_r \cos\theta - \frac{1}{r} U_\theta \sin\theta \\ u_y &= U_r \sin\theta + \frac{1}{r} U_\theta \cos\theta \\ \Delta u &= U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta}. \end{aligned}$$

3. a) Consider the two-dimensional eigenvalue problem on a unit disk,

$$\begin{cases} \Delta u_0 + \lambda_0 u_0 = 0 \text{ inside } B = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \\ u_0 = 0 \text{ on } \partial B. \end{cases} \tag{1}$$

By writing the Laplacian in polar coordinates,

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

show that an eigenfunction u_0 is given by

$$u_0(r, \theta) = J_m(\sqrt{\lambda_0} r)\cos(m\theta), \quad \text{and} \quad u_0(r, \theta) = J_m(\sqrt{\lambda_0} r)\sin(m\theta)$$

where J_m is a Bessel function of m-th order satisfying

$$J_{rr} + \frac{1}{r} J_r - \frac{m^2}{r^2} J + J = 0, \qquad J'(0) = 0$$

and $\sqrt{\lambda_0}$ is a root of $J_m(z)$.

b) Consider the perturbed problem,

$$\Delta u + \lambda u + \varepsilon f(r, \theta) u = 0 \text{ inside } B; \qquad u = 0 \text{ on } \partial B \tag{2}$$

where ε is small. When $\varepsilon = 0$, this problem admits a single eigenvalue $\lambda_0 = 5.783$ corresponding to $m = 0$, where $z = \sqrt{5.783} = 2.4048$ is the first root of $J_1(z)$. How does the perturbation affect this eigenvalue?

c) Consider the eigenvalue problem on a perturbed unit disk

$$\Delta u + \lambda u = 0 \text{ inside } B_\varepsilon; \qquad u = 0 \text{ on } \partial B_\varepsilon \tag{3}$$

where B_ε is the domain whose boundary ∂B_ε is given by $\{r = 1 + \varepsilon f(\theta), \quad \theta = 0 \ldots \pi\}$. When $\varepsilon = 0$, this problem admits a double eigenvalue $\lambda_0 = 14.682$ corresponding to $m = 1$, where $z = \sqrt{14.682} = 3.832$ is the first root of $J_1(z)$. How does the perturbation B_ε of unit disk B affect this eigenvalue?