MATH 4220/5220, Homework 3

Due date: 23 October (Tuesday)

1. 5.1.4
2. 5.3.4, 5.3.8, 5.3.13,
3. 5.4.2, 5.4.12, 5.5.2
4. Consider a unit sphere in 3D with an initially homogeneous temperature distribution $u = 1$. It is then plunged into a cold bath with an outside temperature of 0. This situation is modelled by the following PDE system:
$$\begin{cases} u_t = u_{rr} + \frac{2}{r}u_r, \quad 0 < r \leq 1 \\ u_r(0,t) = 0, \quad u(1,t) = 0 \\ u(r,0) = 1. \end{cases} \tag{1}$$
   (a) Use the separation of variables to write the solution to (1) as a series of the from
   $$u = \sum T_n(t) X_n(r)$$
   where X_n satisfies the ODE
   $$X_{rr} + \frac{2}{r}X_r + \lambda X = 0$$
   for some eigenvalue λ. What boundary conditions must X_n satsify?

   (b) Solve the resulting eigenvalue problem from $X_n(r)$. Hint: Try the following anzatz: $X = \frac{1}{r}\sin(ar)$ for some constant a.

   (c) Write down the full series solution to (1).
5. Let $f(x) = x$. Consider four different representations of $f(x)$:

   (1) $f_1(x) = \sum b_n \sin(nx), \; , \quad x \in (0, \pi)$;

   (2) $f_2(x) = \sum a_n \cos(nx), \quad x \in (0, \pi)$;

   (3) $f_3(x) = \sum a_n \cos(nx) + \sum b_n \cos(nx), \quad , \quad x \in (-\pi, \pi)$;

   (4) $f_4(x) = \sum b_n \sin(\left(n + \frac{1}{2}\right) x), \quad x \in (0, \pi)$.

   Without computing the coefficients, draw how $f_i(x), i = 1..4$ looks like if the series is extended on the whole real line. What is the value of each series at $x = 0$ and at $x = \pm\pi$?
6. Find an example of a sequence of continuous functions that converges pointwise to 0 on the interval $[0, 1]$ but does not converge uniformly.