MATH 5250/4250, Homework 5

Due: Thursday, 13 Nov

1. Consider the system

$$u'' = u - u^p,\ p > 1, \quad -\infty < x < \infty.$$
$$u, u' \to 0 \text{ as } |x| \to \infty$$

(a) Sketch the phase plane (u, v) of the corresponding system $u' = v, v' = u - u^p$.

(b) Determine $u(0)$ without computing $u(x)$.

(c) Determine $u(x)$ by using the anszatz $u(x) = [a\,\text{sech}\,(bx)]^c$ for some numbers a, b, c. Verify that when $p = 2$, $u(x) = \frac{3}{2}\,\text{sech}^2\,(x/2)$. Also verify that $u(0) = a^c$ agrees with $u(0)$ you found in part (b).

2. Let $u_h(z) = \frac{3}{2}\,\text{sech}^2\,(z/2)$, the homoclinic solution to

$$u_{zz} - u + u^2 = 0.$$

(a) Show that $u_h(z) \sim 6e^{-|z|}$ as $|z| \to \infty$.

(b) Consider the system

$$u_{xx} - u + (1 + \varepsilon x^2)u^2 = 0, \quad u'(0) = au(0), \quad u, u' \to 0 \text{ as } x \to \infty.$$

There is a solution of the form $u(x) \sim u_h(x - x_0)$, as $\varepsilon \to 0$, with $x_0 \gg 0$. Determine the value of x_0 as a function of a. Is there any restriction on a?

3. (a) Consider an ODE

$$u_{yy} + f(u) \tag{1}$$

where

$$f(u) = -2(u-1)(u+1)u. \tag{2}$$

Sketch the phase plane of the corresponding system $u' = v, v' = -f(u)$. By direct computation, show that

$$u_\pm(y) = \pm \tanh(y)$$

satisfies (1). Indicate the orbit $u_\pm$ on your phase plane.

(b) Consider the PDE in two dimensions,

$$u_t = \varepsilon^2 \Delta u + f(u) + \varepsilon g(u) = 0, \quad x \in \mathbb{R}^2.$$

where $f(u)$ is as given in (1) and

$$\int_{-1}^{1} g(u)\, du = 1.$$

Find a travelling wave solution of the form

$$u(x) = u_\pm\left(\frac{r - r_0(\varepsilon^p t)}{\varepsilon}\right), \ r = |x|.$$

where $u_\pm$ given by (2) is the heteroclinic solution to (1) [hint: since this solution in radial, use $\Delta u = u_{rr} + \frac{1}{r} u_r$. What value of p should you use? Using solvability conditions, derive an ODE for r_0. Evaluate all the constants in the ODE as much as possible.

(c) Under what conditions does the ODE you derived in part (b) have an equilibrium point? Is that equilibrium stable or unstable?