Math 4/5250
Homework 6, due Tuesday, 25 Nov

1. [Holmes #6, chap. 3.3] The equation for what is known as the Rayleigh oscillator is

$$u_{tt}-\varepsilon\left(1-\frac{1}{3}\left(u_t\right)^2\right)u_t+u=0,\quad u(0)=1;\ u'(0)=0.$$

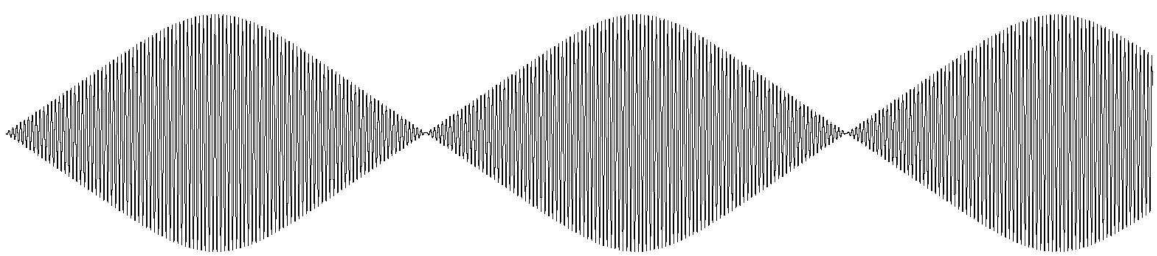
Find the multiple-scales solution valid for large t.

2. [Holmes, #4, chap. 3.4] The equation for a pendulum with a weak forcing is

$$\theta''+\frac{1}{\varepsilon}\sin\left(\varepsilon\theta\right)=\varepsilon^2\sin\left(\omega t\right),\quad t>0;\quad \theta(0)=0=\theta'(0).$$

a) For fixed $\omega^2\neq 1$, find a multiple-scales approximation for $\theta\left(t\right)$.

b) For $\omega=1+\omega_0\varepsilon^2$, find a multiple-scales approximation of $\theta\left(t\right)$.



c) The figure above is the numerical solution with $\varepsilon=0.15$, $\omega=1+\varepsilon^2$, $\theta\left(0\right)=0=\theta'(0)$ for $t\in[0,3000]$, $\theta\in[-3,3]$. Compare the amplitude/phase equations you obtained in part (b) with numerical simulation of the full system. You should include a graph that superimposes the envelope calculations with the full numerics [see Maple's `display` command to superimpose multiple graphs].

3. Consider the system

$$\begin{cases} x'=-x+ay+x^2,\\ y'=-2x+ay.\end{cases}$$

(a) Show that zero equilibrium undergoes a Hopf bifurcation when $a=1$.

(b) Eliminate y,y' to show that $x(t)$ satisfies a second order ODE

$$x''=-ax+x'\left(a-1\right)+2xx'-ax^2. \tag{1}$$

(c) Rescale $x=\varepsilon u(t)$ and let $a=1+\varepsilon^p$, where p is chosen such that $u(t)=O(1)$. Show that to obtain a bounded solution, one choose have $p=2$.

(d) Use the method of multiple scales (or Lindstead method) on to determine the amplitude of $u(t)$. Comment on what this says about the behaviour of the original system near the Hopf bifurcation.

4. Nicholson's blowflies equation is given by

$$\frac{dy}{dt} = ay\left(t-1\right)\exp\left(-y\left(t-1\right)\right) - by$$

with $a, b > 0$. Determine the nonzero constant state and analyse its linear stability. In the (a, b) plane, sketch the stability region.