Math 3110 Homework 7

Due: No later than 6 December, 5pm.

1. [Holmes, 4.2 #7] Consider the eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - r(x)\,y = -\lambda^2 q(x)\,y, \quad y(0) = y(1) = 0. \tag{1}$$

   Here p, q, r are given positive functions and $\lambda > 0$ is the eigenvalue.

   (a) Make a change of variables $y(x) = h(x)\,w(x)$ to transform (1) into

$$p(x)\,w'' + \left[\lambda^2 q(x) - f(x)\right] w = 0$$

   for an appropriate choice of $h(x)$ and $f(x)$.

   (b) Use a WKB approximation to show that for the large eigenvalues, $\lambda \sim \frac{n\pi}{\kappa}, \quad n \to \infty$ where $\kappa = \int_0^1 \sqrt{\frac{q(x)}{p(x)}}dx$. What is the corresponding WKB approximation of the eigenfunctions?

2. [Holmes, #4, chap. 6.5] Consider the following model that describes vibrations in orgain pipes and other such systems (Rayleigh, 1883):

$$\varepsilon y'' - \left(1 - \frac{1}{3}(y')^2\right) y' + y = 0, \quad t > 0.$$

   The numerical solution with initial conditions $y(0) = 0, \quad y'(0) = -\sqrt{3}$ as well as the plot of $v = y'$ vs. y is shown in the figure below. Note that the solution "jumps" from the lower branch to the upper branch of the curve $y = \left(1 - \frac{1}{3}v^2\right)v$ near the fold point $y = -2/3, v = -1$ of that curve. However numerically, we observe that this transition occurs when y is somewhat less than $-2/3$. Analyse this phenomenon and describe the size of this delay as a function of ε. Include a figure superimposing your asymptotic results with direct numerical simulations of (1) for $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

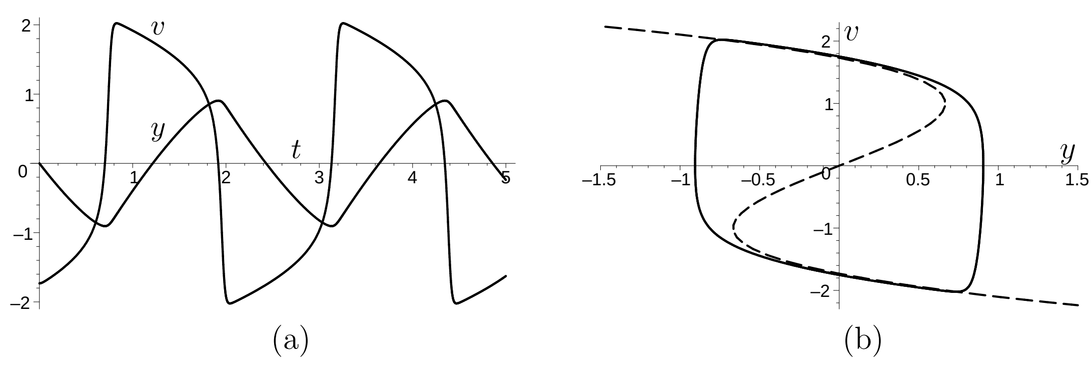

Figure 1: (a) Plot of the solution to (1) with initial conditions $y = 0, v = -\sqrt{3}$ and with $\varepsilon = 0.05$. (b) Phase plot of the same solution in the y, v plane (solid curve), superimposed on the graph of $y = \left(1 - \frac{1}{3}v^2\right)v$ (dashed curve).