WKB Theory & Turning points

Consider: $\varepsilon^2 y'' - q(x) y = 0$

- If $q(x) = q > 0$ is const. then $y = e^{\pm \frac{\sqrt{q}}{\varepsilon} x}$

  $\Rightarrow$ rapid decay / growth

  $\Rightarrow$ $y \sim 0$ is ~~a~~ valid approximation to leading order outer sol'n

- If $q(x) = q < 0$ then $y = e^{\pm i \sqrt{-q}\, x}$

  $\Rightarrow$ Sol'n is <u>oscillatory</u>

  - In this case, $y \sim 0$ is a horrible approximation.

To resolve this issue, we make a WKB Ansatz:

$$y = e^{\frac{\theta(x)}{\varepsilon}} Y(x)$$

$$\Rightarrow \quad y' = e^{\frac{\theta}{\varepsilon}} \left(Y' + \frac{\theta'}{\varepsilon} Y \right)$$

$$y'' = e^{\frac{\theta}{\varepsilon}} \left(Y'' + \frac{2\theta'}{\varepsilon} Y' + \frac{\theta'^2}{\varepsilon^2} Y + \frac{\theta''}{\varepsilon} Y \right)$$

Now expand $Y = y_0 + \varepsilon y_1 + \dots$

We obtain:
$$\begin{cases} \theta_x^2 = q & (O(1)) \\ \theta'' y_0 + 2\theta_0' y_0' + \theta_x^2 y_1 = q y_1 & (O(\varepsilon)) \end{cases}$$

Now $O(1)$ yields: $\theta = \pm \int \sqrt{q}$

and $O(\varepsilon)$ becomes: $\theta'' y_0 + 2\theta_0' y' = 0$

$\Rightarrow$ $y_0 = \dfrac{c}{\sqrt{\theta_x}}$. Combining, we get:

$$(*) \quad y \sim q(x)^{-\frac{1}{4}}\left(a_0 e^{-\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds} + b_0 e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds}\right)$$

Remark: • If $q > 0$ then y has exp. growth and decay

• If $q < 0$ then $\sqrt{q} = i\sqrt{-q}$

and we can rewrite $(*)$ as

$$y \sim \left(-q(x)\right)^{-\frac{1}{4}}\left(\hat{a}_0 \cos\frac{\theta(x)}{\varepsilon} + \hat{b}_0 \sin\left(\frac{\theta(x)}{\varepsilon}\right)\right),$$

$$\theta(x) = \int \sqrt{-q(x)}\,dx$$

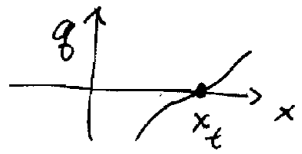
So in this case, y has rapid oscillations.

Turning points: $\varepsilon^2 y'' - q(x)\, y = 0,$

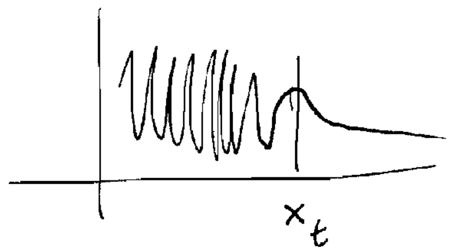
We suppose that $q = 0$ at some $x = x_t$

with $q < 0$, $x < x_t$ and $q > 0$, $x > x_t$; $q'(x_t) > 0$

Then



- y oscillates very fast, $x < x_t$
- y decays very fast, $x > x_t$
  (grows)



We write:

$$y \sim \begin{cases} y_L, & x < x_t \\ y_R, & x > x_t \end{cases}$$

$$y_R = q(x)^{-\frac{1}{4}} \left(a_R\, e^{-\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds} + b_R\, e^{-\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds} \right)$$

$$y_L = q(x)^{-\frac{1}{4}} \left(a_L\, e^{-\frac{i}{\varepsilon}\int_{x}^{x_t} \sqrt{q}} + b_L\, e^{-\frac{i}{\varepsilon}\int_{x}^{x_t} \sqrt{q}} \right)$$

Q: Find a_R, b_R in terms of a_L, b_L (or vice-versa)

Transition layer: Near $x = x_t$, we rescale:

$$x = X_t + \varepsilon^p z, \quad y = Y$$

$$q(x) \sim \varepsilon^p z\, q'(x_t)$$

<u>We get</u>: $\varepsilon^{2-2p} Y'' - q'(x_t) z Y \varepsilon^{p} \approx 0$

<u>Balance</u>: $2 - 2p = p \Rightarrow \boxed{p = \frac{2}{3}}$

<u>Scale out $q'(x_t)$</u>: $z = \left(q'(x_t)\right)^{-\frac{1}{3}} s$

$$\Rightarrow \quad Y_{ss} - sY = 0$$

$$\Rightarrow \quad Y \sim a\, A_i\left(\left(q'(x_t)\right)^{\frac{1}{3}} z \right) + b\, B_i\left(q'(x_t)^{\frac{1}{3}} z \right)$$

where A_i, B_i are Airy fcs, a, b are to be found through matching.

<u>Matching</u>: A_i, B_i are given by:

$$A_i(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}\, |x|^{\frac{1}{4}}} \cos\left(\frac{2}{3} |x|^{3/2} - \frac{\pi}{4} \right), & x \to -\infty \\[2ex] \dfrac{1}{2\sqrt{\pi}\, x^{\frac{1}{4}}}\, e^{-\frac{2}{3}|x|^{3/2}}, & x \to +\infty \end{cases}$$

$$B_i(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}\, |x|^{\frac{1}{4}}} \cos\left(\frac{2}{3} |x|^{3/2} + \frac{\pi}{4} \right), & x \to -\infty \\[2ex] \dfrac{1}{\sqrt{\pi}\, x^{\frac{1}{4}}}\, e^{+\frac{2}{3}|x|^{3/2}}, & x \to \infty \end{cases}$$

For $x > x_t$: $\quad z = \dfrac{x - x_t}{\varepsilon^p}$, $p = \dfrac{2}{3}$

Outer : $\displaystyle\int_{x_t}^{x} \sqrt{q(s)}\, ds \approx \varepsilon^p \int_0^{z} \sqrt{\hat{s}\, \varepsilon^p q' \cancel{z}}\, d\hat{s} \sim \frac{2}{3}\varepsilon\, q'^{\frac{1}{2}} z^{\frac{3}{2}}$

$s = x_t + \hat{s}\,\varepsilon^p$

$x = x_t + \varepsilon^p z$

$$q^{-\frac{1}{4}} \sim \varepsilon^{-\frac{1}{6}} q'^{-\frac{1}{4}} z^{-\frac{1}{4}}$$

$$\Rightarrow\ y_R \sim \varepsilon^{-\frac{1}{6}} q'^{-\frac{1}{4}} z^{-\frac{1}{4}} \left(a_R\, e^{-\frac{2}{3} q'^{\frac{1}{2}} z^{\frac{3}{2}}} + b_R\, e^{+\frac{2}{3} q'^{\frac{1}{2}} z^{\frac{3}{2}}} \right),$$

Inner : $\displaystyle Y \sim \frac{1}{2\sqrt{\pi}\, q'^{\frac{1}{12}} z^{\frac{1}{4}}} \left(a e^{-\frac{2}{3} q'^{\frac{1}{2}} z^{\frac{3}{2}}} + b\, e^{\frac{2}{3} q'^{\frac{1}{2}} z^{\frac{3}{2}}} \right)$ $\quad z \to +\infty$

$$\Rightarrow\ \boxed{\begin{aligned} a_R &= \frac{\varepsilon^{\frac{1}{6}} q'^{\frac{1}{6}}}{2\sqrt{\pi}}\, a \\ b_R &= \frac{\varepsilon^{\frac{1}{6}} q'^{\frac{1}{6}}}{\sqrt{\pi}}\, b \end{aligned}}$$

For $x < x_t$: To avoid complex numbers,

rewrite : $\displaystyle y_L = \frac{1}{|q(x)|^{\frac{1}{4}}} \left(\hat{a}_L \cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right) + \hat{b}_L \cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right) \right)$

Where $\displaystyle\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}\, ds$

Performing matching, we find

$$\hat{a}_L = \frac{\varepsilon^{\frac{1}{6}} q'^{\frac{1}{6}}}{\sqrt{\pi}} = 2 a_R$$

$$\hat{b}_L = \frac{\varepsilon^{\frac{1}{6}} q'^{\frac{1}{6}}}{\sqrt{\pi}} b = b_R$$

<u>In summary:</u>

$$y \sim \begin{cases} \frac{1}{|q(x)|^{\frac{1}{4}}} \left(2 a_R \cos\left(\frac{1}{\varepsilon} \theta(x) - \frac{\pi}{4} \right) + b_R \cos\left(\frac{1}{\varepsilon} \theta(x) + \frac{\pi}{4} \right) \right), & x < x_t \\ \frac{1}{|q(x)|^{\frac{1}{4}}} \left(a_R e^{-\frac{1}{\varepsilon} K(x)} + b_R e^{\frac{1}{\varepsilon} K(x)} \right), & x > x_t \end{cases}$$

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}\, ds, \quad K(x) = \int_{x_t}^{x} \sqrt{q(s)}\, ds$$

<u>Homework</u>: Verify that the uniform sol'n is given by:

$$\begin{cases} y \sim \frac{K(x)}{|q(x)|^{\frac{1}{4}}} \left(a_0 A_i\left(\left(\frac{3 K(x)}{2 \varepsilon} \right)^{\frac{2}{3}} \right) + b_0 B_i\left(\left(\frac{3 K}{2 \varepsilon} \right)^{\frac{3}{2}} \right) \right) \\ a_0 = 2 \sqrt{\pi} \left(\frac{3}{2 \varepsilon} \right)^{\frac{1}{6}} \quad b_0 = \sqrt{\pi} \left(\frac{3}{2 \varepsilon} \right)^{\frac{1}{6}} \end{cases}$$

Wave Propagation: Consider the equation for displacement of a spring:

$$\begin{cases} u_{xx} = \mu^2(x)\, u_{tt} + \alpha(x)\, u_t + \beta(x) u \,, & \begin{matrix} 0 < x < \infty \\ t > 0 \end{matrix} \\ u(0,t) = \cos(\omega t) \end{cases}$$

- αu_t is the damping term
- βu is elastic support
- $\cos(\omega t)$ is periodic forcing at the left end.
- If $\alpha = \beta = 0$ then we make an anzatz: $u = e^{it\omega - kx}$ $\rightarrow k = \pm \omega \mu$
- More generally consider the limit of large frequency, $\boxed{\omega \to \infty}$ and expand using WKB-like anzatz:

$$u = e^{i(\omega(t - \theta(x))}\left[u_0 + \frac{1}{\omega} u_1 + \dots \right]$$

We get: $\begin{cases} \theta_x^2 = \mu^2 \\ \theta_{xx} u_0 + 2\theta_x u_{0x} = -\alpha u_0 \end{cases}$

Solving, we get $\theta = \pm \int_0^x \mu(s)\,ds$

and $u_0 = a_0 e^{\int \frac{-\alpha - \theta_{xx}}{2\theta_x}} = \frac{a_0}{\sqrt{\mu}} e^{-\frac{1}{2}\int \frac{\alpha}{\mu}}$

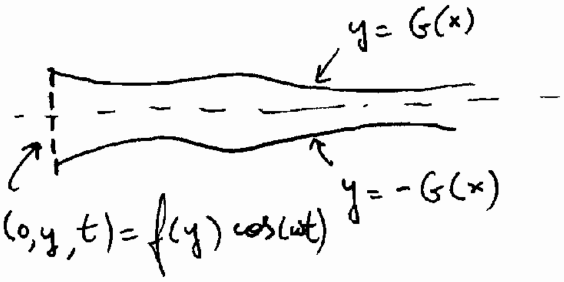
Imposing boundary condition $u(0,t) = \cos(\omega t)$ we then get:

$$u(x,t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}} \exp\left(-\frac{1}{2} \int_0^x \frac{\alpha(s)}{\mu(s)}\,ds\right) \cos\left(\omega t - \omega \int_0^x \mu(s)\,ds\right)$$

- Note that the amplitude is decreasing as $\alpha(x)$ is increased.
- Leading order is indep. of β.

Wave propagation in thin membrane

Consider an elastic membrane that is fixed along lateral sides and is very long, being forced periodically at the left end $x=0$:



Model:

$$\varepsilon^2 u_{xx} + u_{yy} = \mu^2(x)\, u_{tt} \quad , \quad \begin{cases} 0 < x < \infty \\ -G(x) < y < +G(y) \end{cases}$$

$$u(x,y,t) = 0 \quad , \quad y = \pm G(x)$$

$$u(0,y,t) = f(y)\cos(\omega t)$$

- Forcing sends waves down the membrane
- We will see that these waves can get "stuck".

Travelling-wave solution anzatz:

$$u(x,y,t) \sim e^{i(\omega t - \theta(x)/\varepsilon)} \left(u_0(x,y) + \varepsilon u_1(x,y) + \dots \right)$$

$$\Rightarrow \quad O(1): \begin{cases} u_{0\,yy} + (\omega^2\mu^2 - \theta_x^2)\, u_0 = 0 \qquad (*) \\ u_0 = 0 \quad \text{when} \quad y = \pm G(x) \end{cases}$$

$$O(\varepsilon): \begin{cases} u_{1\,yy} + (\omega^2\mu^2 - \theta_x^2)\, u_1 = i\left(\theta_{xx} u_0 + 2\theta_x u_{0x} \right) \qquad (**) \\ u_1 = 0 \quad \text{when} \quad y = \pm G(x) \end{cases}$$

From $O(1)$ we get: $u_0 = A(x)\sin\left(\lambda_n(y+G)\right)$

where $\lambda_n^2 = \omega^2\mu^2 - \theta_x^2$

Moreover, $\boxed{2G\lambda_n = n\pi}$

So we get: $\theta_x = \pm\sqrt{\omega^2\mu^2 - \lambda_n^2}$, $\lambda_n = \frac{n\pi}{2G}$, $n=1,\ldots$

- Two sol'n "$\pm$" corresponding to incoming or outgoing waves
- If n is large enough so that $n \geq \frac{2\omega G(x)\mu(x)}{\pi}$ then propagation terminates for that mode.

To determine $A(x)$, we apply a solvability condition to (**):

$$\underbrace{\int_{-G}^{G} u_0\left(u_{1yy} + \lambda_n^2 u_1\right)dy}_{0} = i\int_{-G}^{G} \underbrace{u_0^2\,\theta_{xx} + 2\theta_x u_{0x}u_0}_{\frac{\partial}{\partial x}\left(\theta_x u_0^2\right)}\,dy$$

Note that $\frac{d}{dx}\int_{-G}^{G} f(x;y)\,dy = G' f\Big|_{-G}^{G} + \int_{-G}^{G}\frac{\partial}{\partial x} f\,dy$

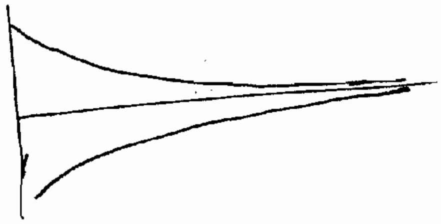
So we get: $\frac{d}{dx}\underbrace{\int_{-G}^{G}\theta_x u_0^2\,dy}_{\theta_x \int_{-G}^{G} u_0^2 \;=\; a} = \left(G'\theta_x \cancel{u_0^2}\Big|_{-G}^{G}\right) \to 0$

$\Rightarrow$ $\boxed{A(x) = \frac{a}{\sqrt{\theta_x G(x)}}}$

This sol'n is valid provided that $n \geqslant \frac{2\omega}{\pi} G(x)\mu(x)$

In particular, suppose that G looks like

with $G \to 0$ as $x \to 0$
and $\mu = \text{const.}$



Then $\exists\ x_t$ where

$\theta_x = 0$,

This is a <u>turning point</u>.

Waves of mode $\geqslant n$ will not propagate beyond x_t.

Near $x = x_t$ we can apply turning-point theory using a scaling $x = x_t + \varepsilon^{\frac{2}{3}} \overline{X}$ as before.

In the end, it is possible to obtain the following <u>connection formulae</u> [for details, see Holmes, §4.5]

$$u(x,y,t) \sim \frac{a_n}{\sqrt{\theta_x\, G(x)}}\, V_n(x) \sin\big(\lambda_n (y + G)\big) \cos(\omega t)$$

where

$$V_n(x) = \begin{cases} \sin\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right), & 0 < x < x_t \\ e^{-\varphi(x)/\varepsilon}, & x > x_t, \end{cases}$$

$$\theta = \int_x^{x_t} \sqrt{\omega^2\mu^2 - \lambda_n^2}\,, \qquad \varphi = \int_{x_t}^{x} \sqrt{\lambda_n^2 - \omega^2\mu^2}$$

W5

- The constants a_n are obtained from initial conditions.
- The incoming wave is (partially) reflected at $x = x_t$ back to the origin and the part that ~~at~~ gets through decays exponentially for $x > x_t$
- $x_t(n)$ moves to left as ω is decreased. If ω is sufficiently decreased, x_t becomes 0. This is called <u>cutoff frequency</u>.

Reference:
Holmes, perturbation methods

Delayed bifurcations:

Consider a bifurcation pbm,

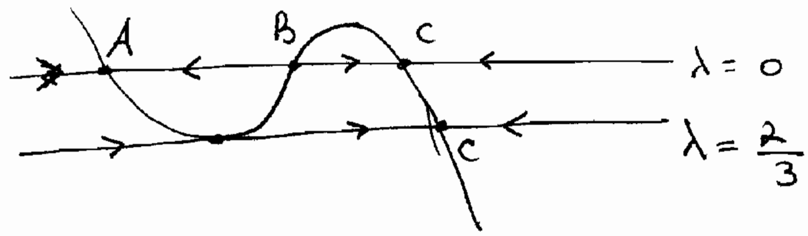
$$y' = f(y, \lambda)$$

Suppose that $\lambda = \lambda_c$ is a bifurcation point with

$$f(y_c, \lambda_c) = 0, \quad f_y(y_c, \lambda_c) = 0$$

and $f_{yy} > 0$, $f_\lambda > 0$. For example,

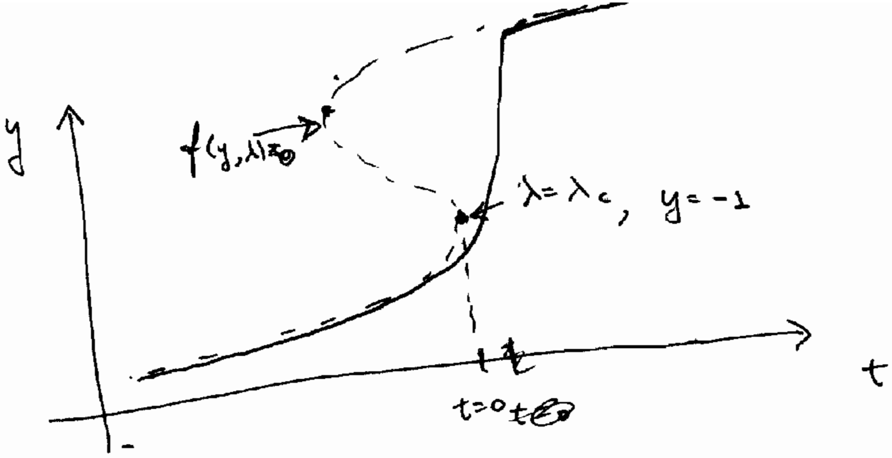
$$f(y, \lambda) = y - \frac{1}{3} y^3 + \lambda$$



- Bifurcation as λ goes through $\lambda_c = \frac{2}{3}$
- 2 steady states $^{(A,B)}$ if $\lambda < \frac{2}{3}$, one unstable s.s. (B)
- 1 steady state if $\lambda > \frac{2}{3}$
- "Sudden jump" in sol'n A is expected as λ crosses λ_c.

Consider what happens if we <u>slowly</u> increase λ.

D2



Set $\lambda = \lambda_c + \varepsilon t$, $\varepsilon \ll 1$ and

<u>expand</u> near $\lambda = \lambda_c$:

$$y = y_c + \varepsilon^p y_1$$

$$f(y,\lambda) = \frac{f_{yy}}{2}(\varepsilon^p y_1)^2 + f_\lambda \varepsilon t = \varepsilon^p y_{1_t}$$

<u>Rescale time</u>: $t = \varepsilon^q \tau$

$$\Rightarrow \quad \varepsilon^{p-q} y_{1\tau} = \varepsilon^{2p} \frac{f_{yy}}{2} y_1^2 + \varepsilon^{1+q} f_\lambda t$$

<u>Balance</u>: $p - q = 2p = 1 + q \Rightarrow p = -q$,

$2p = 1 - p$

$\boxed{p = \frac{1}{3}}$ $\boxed{q = \frac{-1}{3}}$

$$\Rightarrow \quad y_{1\tau} = \frac{f_{yy}}{2} y_1^2 + f_\lambda t$$

Scale further:

$$y_1 = \beta V, \qquad \tau = \delta s$$

to get

$$\boxed{V_s = -V^2 + s} \qquad (*)$$

$$\beta = \left(\frac{4 f_\lambda}{f_{yy}^2}\right)^{\frac{1}{3}}, \qquad \delta = -\left(\frac{2}{f_\lambda f_{yy}}\right)^{\frac{1}{3}}$$

Now consider transformation:

$$V = \frac{\varphi'(s)}{\varphi(s)} \quad \Rightarrow \quad V_s = \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}$$

$$\Rightarrow \quad \varphi''(s) = s\,\varphi(s)$$

$$\Rightarrow \quad V(s) = \frac{a_0 A_i'(s) + a_1 B_i'(s)}{a_0 A_i(s) + a_1 B_i(s)}$$

Now as $s \to \infty$ $[t < 0]$ we get:

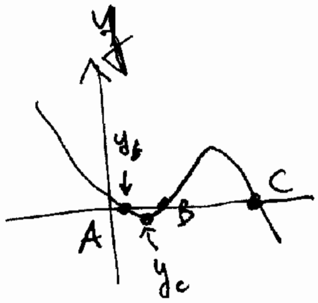
$$V(s) \approx \begin{cases} +\sqrt{s}, & a_1 \neq 0, \quad s \to \infty \\ -\sqrt{s}, & a_1 = 0 \end{cases}$$

[from asymptotics of A_i, B_i]

Matching: if $\underline{t<0}$ then $y_{1_t} \sim 0$

$$\Rightarrow \frac{f_{yy}}{2} y_1^2 + f_\lambda t \sim 0$$

$$y_1 \sim \pm \sqrt{-t \frac{2 f_\lambda}{f_{yy}}}$$



Must take $\ominus$ sol'n since $y_1 < 0$; y is on branch "A" for $\lambda < \lambda_c$

$$\Rightarrow a_1 = 0;$$

$$\boxed{v(s) = \frac{Ai'(s)}{Ai(s)}}$$

$$\Rightarrow \varepsilon^{\frac{1}{3}} y_1(t) = \left(\frac{4 f_\lambda \varepsilon}{f_{yy}^2}\right)^{\frac{1}{3}} \frac{Ai'(s)}{Ai(s)}; \quad \boxed{t = \varepsilon^{-\frac{1}{3}} \left(\frac{2}{f_\lambda f_{yy}}\right)^{\frac{1}{3}} (-s)}$$

Now this sol'n $\underline{\text{breaks down}}$ at $s = s_0 = -2.33$ where s_0 is the first root of $Ai(s_0) = 0$

$\underline{\text{Near}}$ $s = s_0$, we have the asymptotics:

$$y - y_c \sim \left(\frac{4 f_\lambda \varepsilon}{f_{yy}}\right)^{\frac{1}{3}} \frac{1}{s - s_0}, \quad s \to s_0$$

The breakdown occurs at the time

$$t \sim \varepsilon^{-\frac{1}{3}} \left(\frac{2}{f_\lambda f_{yy}}\right)^{\frac{1}{3}} \cdot 2.33$$

Example:

$$y' = y - \frac{1}{3}y^3 + \lambda,$$

take $\lambda = \frac{2}{3} + \varepsilon t$, $t = -100 \ldots +100$,

$\varepsilon = 0.01$, $y(0) = -2$.

Then we have $f(y,\lambda) = y - \frac{1}{3}y^3 + \lambda$

$\lambda_c = \frac{2}{3}$, $y_c = -1$

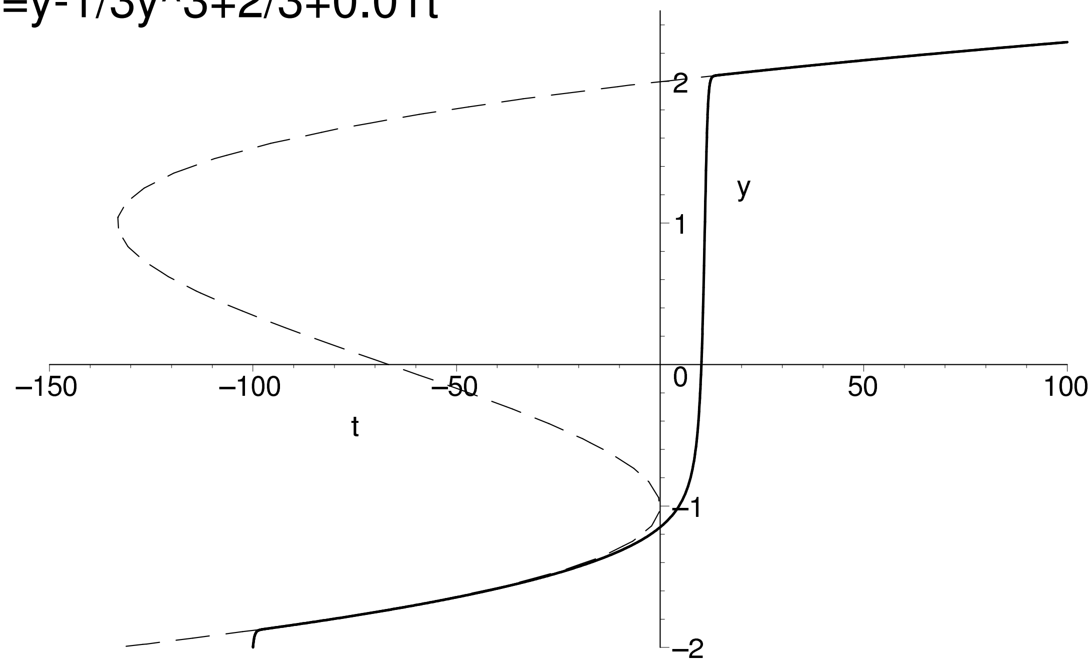
$f_\lambda = 1$, $f_{yy} = 2$, $\left(\frac{2}{f_\lambda f_{yy}}\right)^{\frac{1}{3}} = 1$

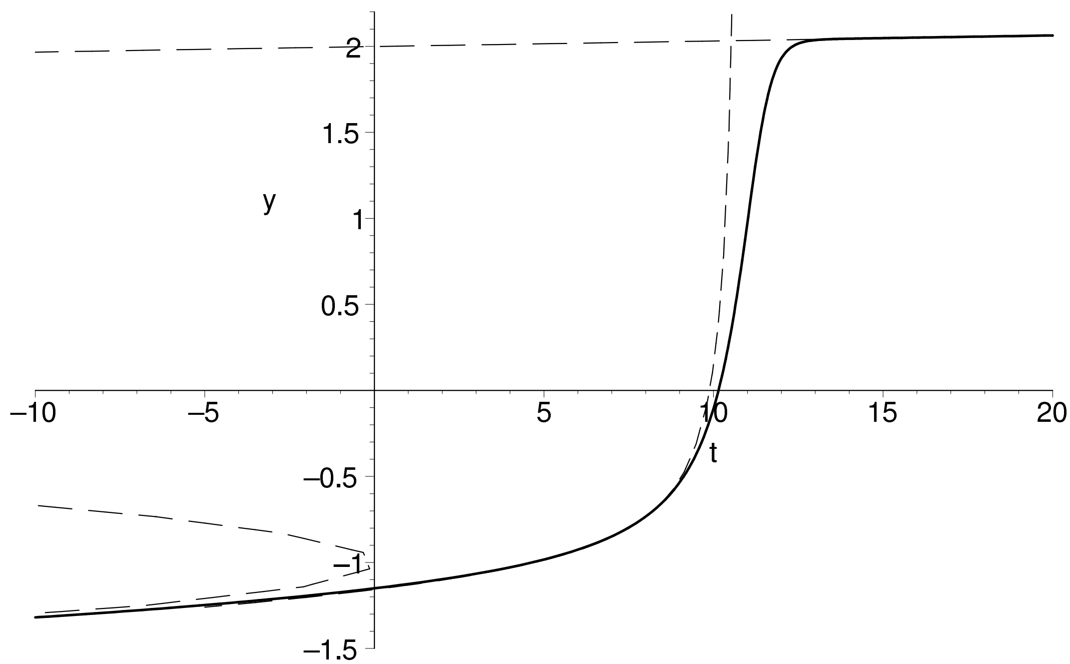
So breakup occurs at $t \sim 2.33 \cdot \varepsilon^{-\frac{1}{3}}$.

If $\varepsilon = 0.01$ then $2.33 \times 0.01^{-\frac{1}{3}} \sim 10.8$

See figure on the next page.

Dashed curve is the slow-state solution f=0;
solid curve is the numerical solution to
y'=y-1/3y^3+2/3+0.01t





Blowup of the top figure. Dashed curve shows the asymptotic estimate y ~ -1+0.01^(1/3) y1. Note the transition delay of about 10, as predicted.