Homework 1 (Kolokolnikov)

Due: Tuesday

1. The Van der Pol oscillator is governed by the ODE

$$u_{tt} + \varepsilon u_t \left(u^2 - 1\right) + u = 0. \tag{1}$$

   (a) Using the method of multiple scales, show that the leading order expansion is of the form

$$u = A(\varepsilon t) \cos\left(t + \Phi\left(\varepsilon t\right)\right)$$

   and find the equations for A and Φ.

   (b) Solve for A and Φ subject to initial conditions $u\left(0\right) = 1, \quad u_t\left(0\right) = 0$.

   (c) Use Maple to plot a numerical solution to (1) for $\varepsilon = 0.1$, $t \in (0, 100)$. On the same graph, also plot the envelope $A\left(\varepsilon t\right)$ (hint: see website for sample maple worksheet to do this). Comment on what you observe.

2. This problem demonstrates how the method of multiple scales can be used for studying periodic solutions resulting from a Hopf bifurcation. Consider the system

$$\begin{cases} x' = -x + ay + x^2, \\ y' = -2x + ay. \end{cases}$$

   (a) Show that the zero equilibrium $x = 0, y = 0$ undergoes a Hopf bifurcation when $a = 1$; that it is stable when $a < 1$ and unstable when $a > 1$.

   (b) Eliminate y, y' to show that $x(t)$ satisfies a second order ODE

$$x'' = -ax + x'\left(a - 1\right) + 2xx' - ax^2. \tag{2}$$

   (c) Rescale $x = \varepsilon u(t)$ and let $a = 1 + \varepsilon^2$ to obtain

$$u_{tt} + u = \varepsilon\left(-u^2 + 2uu_t\right) + \varepsilon^2\left(u_t - u\right) + O\left(\varepsilon^3\right). \tag{3}$$

   (d) Apply the method of multiple scales to (3) to determine the envelope of the oscillations.

   (e) What is the amplitude of $u(t)$ for large t?

   (f) Use Maple to plot a numerical solution to (3) for $\varepsilon = 0.1$, $t \in (0, 100)$ and using initial conditions $u(0) = 1, u'(0) = 0$. On the same graph, also plot the envelope. Comment on what you observe.

   (f) Take $a = 1.04$. Integrate (2) numerically until you get a periodic cycle. Then compute $M(a) = \max(x) - \min\left(x\right)$ numerically[1]. Then use part (e) to provide an analytical estimate of $M(a)$. How big is the relative error?

   (g) Based on your result in (e), sketch the theoretical prediction of the graph of $M(a)$ as a function of a, with a close to 1 (with a either to the left or to the right of 1).

3. A model of a laser subject to opto-electronic feedback is desribed by the following system [Erneux]:

$$x' = -y - \eta(1 + y(s - \theta)); \qquad y' = (1 + y)x. \tag{4}$$

   The parameters η and θ represent feedback parameter and delay time, respectively.

   (a) Linearize around the steady state $x = 0$, $y = -\eta/(1 + \eta)$. What is the transcendental equation for the resulting eigenvalue λ?

[1] More precisely, $M\left(a\right) = \limsup_{t\to\infty} x(t) - \liminf_{t\to\infty} x(t)$.

(b) Seek Hopf bifurcations, i.e. plug in $\lambda = i\omega$, then separate real and imaginary parts. Show that Hopf bifurcations occur when

$$\sin(\omega\theta) = 0 \quad \text{or} \quad (1+\eta)\,\omega^2 = 1 + \eta\cos(\omega\theta)\,. \tag{5}$$

(c) Equations (5) represent curves in the (η, θ) plane. Plot these curves. Analytically classify any intersection points that you may observe. The first such point occurs when $\theta = 2\pi$, $\eta = 3/5$. What are the possible values of ω at this point? List at least three more such intersections.

(d) [BUNUS MARKS] These intersection points are called double-hopf points and very interesting dynamics can be observed near these points. Perform numerical experiments of (4) with θ near 2π, and with η near $3/5$. Play around with η and θ near these values and describe what you observe. Hand in any useful plots of numerics. REMARK: You can use matlab for solving delay ode's, see `dde23` command. Or you can use the free "Dynamics Solver" program (google it). If in doubt, ask me.