2) $$u'' + u = \varepsilon(-u^2 + 2uu') + \varepsilon^2(u' - u)$$

$$u = \mathcal{U}(t, \tau, s) \qquad \tau = \varepsilon t, \quad s = \varepsilon^2 t$$

$\mathcal{U} = \mathcal{U}_0 + \varepsilon \mathcal{U}_1 + \dots$ As in q. 2, $\mathcal{U}_0(t,\tau,s) = \mathcal{U}_0(t,s)$

$$\mathcal{U}_0 = A(s)e^{it} + c.c.$$

$$\mathcal{U}_{1tt} + \mathcal{U}_1 = -\mathcal{U}_0^2 + \underbrace{2\mathcal{U}_0\mathcal{U}_{0t}}_{(\mathcal{U}_0^2)_t} \qquad \cancel{-2\mathcal{U}_{0t\tau}}, \text{ resonance, } = 0$$

$$= -A^2e^{2it} - \bar{A}^2e^{-2it} - 2A\bar{A} + 2iA^2e^{2it} - 2i\bar{A}^2e^{-2it}$$

$$= e^{2it}(A^2)(-1+2i) + c.c. - 2A\bar{A}$$

$$\to \mathcal{U}_1 = A^2\frac{(1-2i)}{3}e^{2it} + c.c. - 2A\bar{A}$$

$$\mathcal{U}_{2tt} + \mathcal{U}_2 = \mathcal{U}_{0t} - \mathcal{U}_0 - 2\mathcal{U}_0\mathcal{U}_1 + 2\mathcal{U}_0\mathcal{U}_1' + 2\mathcal{U}_1\mathcal{U}_0' - 2\mathcal{U}_{0ts}$$

$$= \mathcal{U}_{0t} - \mathcal{U}_0 + 2\left[-\mathcal{U}_0\mathcal{U}_1 + (\mathcal{U}_0\mathcal{U}_1)'\right] - 2\mathcal{U}_{0ts}$$

$$u_0 u_1 = e^{it}\left(\frac{1-2i}{3}|A|^2A - 2|A|^2A\right) = e^{it}|A|^2A\left(\frac{-5-2i}{3}\right) + \dots$$

$+ \dots$

$$-u_0 u_1 + (u_0 u_1)' = e^{it}|A|^2A\underbrace{\left(\frac{5+2i}{3}\right)(1-i)}_{\frac{1}{3}(\cancel{3}7-3i)}$$

$$u_{2tt} + u_2 = e^{it}\left\{A(-1+i) + \frac{2}{3}|A|^2A(7-3i) - \cancel{2}2iA_s\right\}$$

$+ \dots$

$$\boxed{2iA_s = A(-1+i) + \frac{2}{3}|A|^2A(7-3i)}$$

$$A = Re^{i\varphi}$$

$$2iR_s - 2R\varphi_s = R(-1+i) + \frac{2}{3}R^3(7-3i)$$

$$R_s = R - 2R^3 \Rightarrow R \to \frac{1}{\sqrt{2}} \text{ as } s \to \infty$$

$$u \sim 2R\cos(t+\varphi) \Rightarrow \boxed{u \to \sqrt{2} \text{ as } t \to \infty}$$

3) $x' = -y - \eta(1 + y(s-\theta))$, $y' = (1+y)x$

S.S. $x = 0$, $y = -\eta(1-y) \Rightarrow y_e = -\frac{\eta}{1+\eta}$

Stability: $\begin{cases} x = 0 + e^{\lambda t}\varphi \\ y = y_e + e^{\lambda t}\psi \end{cases}$

$\Rightarrow \lambda\varphi = -\psi - \eta e^{-\lambda\theta}\psi$

$\lambda\psi = \underbrace{(1+y_e)}_{\frac{1}{1+\eta}}\varphi$

$$\lambda^2(1+\eta) = -1 - \eta e^{-\lambda\theta}$$

Hopf: $\lambda = i\omega$: $-\omega^2(1+\eta) = -1 - \eta(\cos\omega\theta - i\sin(\theta\omega))$

$\Rightarrow \sin\omega\theta = 0$, $(1+\eta)\omega^2 = 1 + \eta\cos(\omega\theta)$

Sol'n 1: $\omega\theta = \pi n$, n $\underset{\neq 0}{\text{even}}$ $\Rightarrow (1+\eta)\left(\frac{\pi n}{\theta}\right)^2 = 1+\eta \Rightarrow \boxed{\theta = n\pi}$, $\eta \equiv$ anything, n even

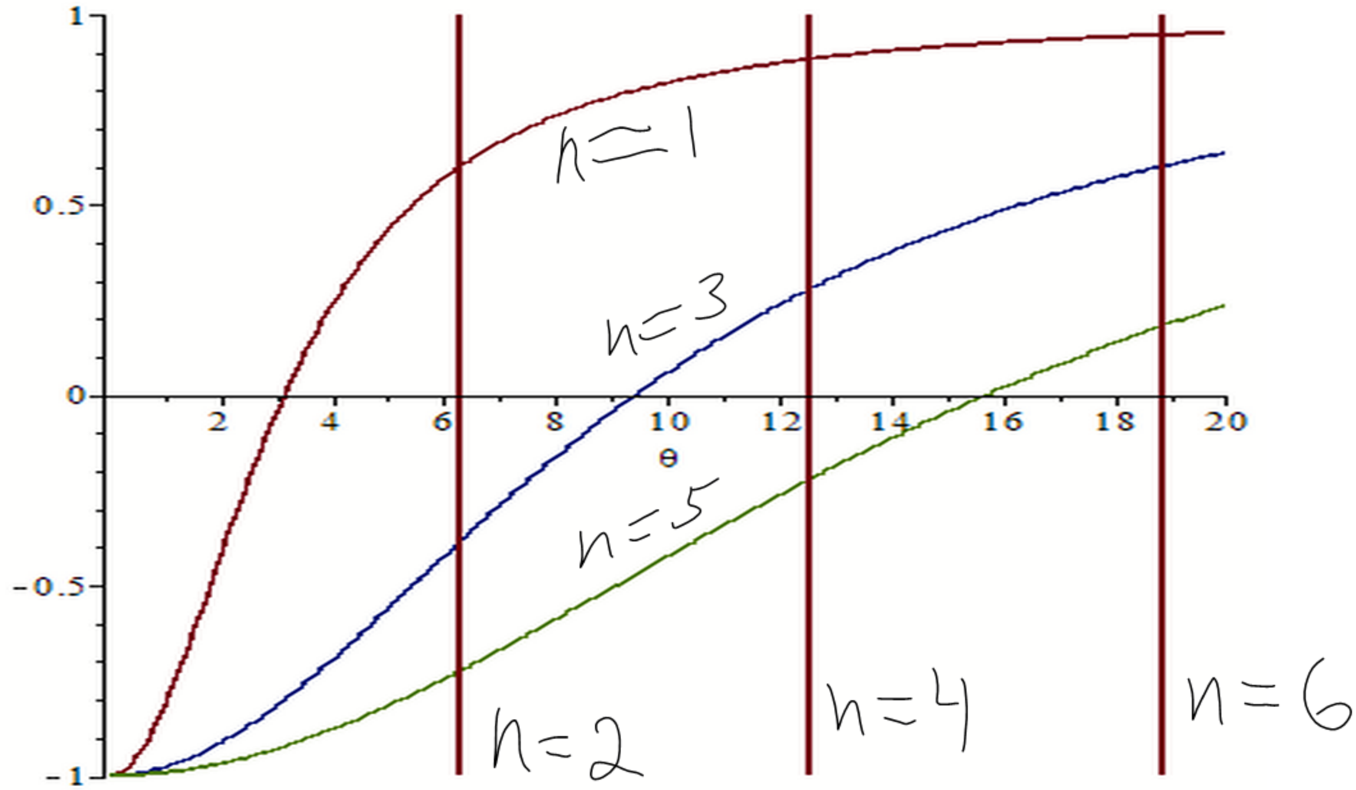
Sol'n 2: $\omega\theta = \pi n$, n odd, $(1+\eta)\left(\frac{\pi n}{\theta}\right)^2 = 1-\eta \Rightarrow \boxed{\eta = \frac{2}{1+\left(\frac{\pi n}{\theta}\right)^2} - 1}$, n odd

Intersections of sol 1 & 2: $\theta = m\pi$, m even,

$\eta = \frac{2}{1+\left(\frac{n}{m}\right)^2} - 1$, n odd

Ex: $m = 2$, $n = 1$, $\theta = 2\pi$,
$\eta = \frac{2}{1+\frac{1}{4}} - 1 = \frac{3}{5}$

Ex: $m = 2$, $n = 3$, $\eta = \frac{2}{1+\frac{9}{4}} - 1 = \frac{-5}{13}$


1
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