Bilinear maps and harmonic functions

A <u>bilinear map</u> is an analytic function of the form

$$w = f(z) = \frac{az+b}{cz+d}, \quad \text{with} \quad ad \neq bc$$

<u>Ex</u>: Consider the map $w = i\,\dfrac{(1-z)}{1+z} = f(z)$.

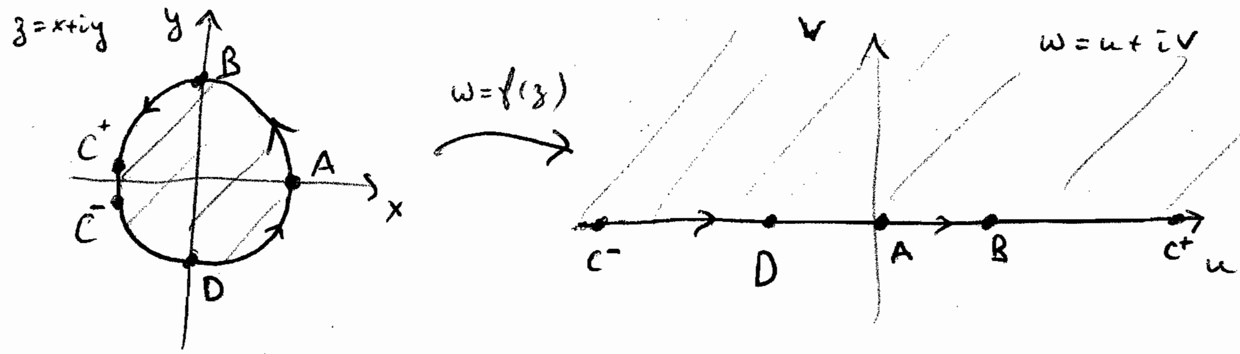
If $z = e^{i\theta}$ is on the unit circle, then

$$f(z) = \frac{i\,(1-e^{i\theta})}{1+e^{i\theta}} = i\left(\frac{e^{-i\theta/2}-e^{i\theta/2}}{e^{-i\theta/2}+e^{i\theta/2}}\right)$$

$$= \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2 \in \mathbb{R}$$

Thus $w = f(z)$ maps the unit circle into x-axis:

$$f(1) = 0, \quad f(i) = \tan\tfrac{\pi}{4} = 1, \quad f(-i) = -1, \quad f(-1) = \pm\infty$$



Moreover it maps the inside of the unit circle onto the upper-half plane [since analytic maps preserve orientation]

In general, bilinear maps takes circles and lines to circles and lines.

Prof: Any bilinear map can be written as a combination of linear maps and $\frac{1}{z}$. To wit, let $L_1(z) = Az+B$, $L_2(z) = Cz+D$, then

$$w = f(z) = L_1 \circ \frac{1}{z} \circ L_2 = A\left(\frac{1}{Cz+D}\right) + B = \frac{A+DB+CBz}{Cz+D}$$

$$= \frac{az+b}{cz+d} \quad \text{by choosing}$$

$$C=c,\ D=d,\ B=\frac{a}{c},\ A=b-\frac{da}{c}.$$

Now linear maps rotate/scale/shift. So a linear map maps circles into circles and lines into lines. It remains to show that $\frac{1}{z}$ also maps circles and lines into circles and lines.

Given a circle $(x-x_0)^2+(y-y_0)^2 = a^2$, let $w = u+iv = \frac{1}{z} = \frac{1}{x+iy}$; then

$$\left\{ u = \frac{x}{x^2+y^2},\ v = \frac{y}{x^2+y^2} \right\} \iff \left\{ x = \frac{u}{u^2+v^2},\ y = \frac{v}{u^2+v^2} \right\}$$

Now $x^2+y^2-2xx_0-2yy_0+x_0^2+y_0^2-a^2=0$

becomes $\frac{1}{u^2+v^2} - 2x_0\frac{u}{u^2+v^2} - 2y_0\frac{v}{u^2+v^2} + x_0^2+y_0^2-a^2 = 0$

or $$u^2+v^2-\frac{2x_0}{x_0^2+y_0^2-a^2}u-\frac{2y_0}{x_0^2+y_0^2-a^2}v+\frac{1}{x_0^2+y_0^2-a^2}=0$$

or $$(u-u_0)^2+(v-v_0)^2=b^2$$

Where $$u_0=\frac{x_0}{x_0^2+y_0^2-a^2}\ ;\ v_0=\frac{y_0}{x_0^2+y_0^2-a^2}\ ;$$

$$b^2=\frac{1}{a^2-x_0^2-y_0^2}+u_0^2+v_0^2 .$$

This is again a circle, provided that $x_0^2+y_0^2-a^2\neq 0$

On the other hand, if $x_0^2+y_0^2-a^2=0$

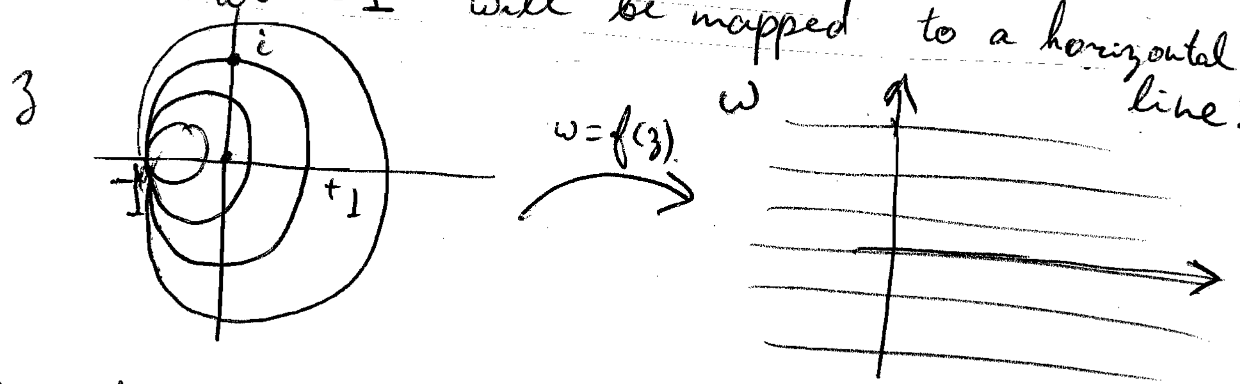
then we get $2x_0u+2y_0v=1$, Which is a line

[in this case, the circle goes through the origin].

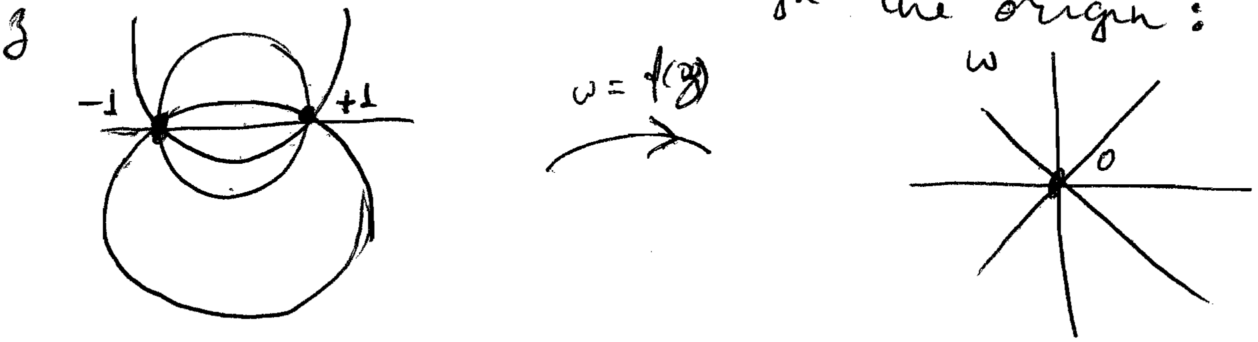
Ex Consider the map $\omega = f(z) = \frac{i(1-z)}{1+z}$.

We have shown that it maps a unit circle onto x-axis. More generally, any circle or line going through $z=-1$ will be mapped into a straight line, since its image must be unbounded as $f(-1)=\infty$.

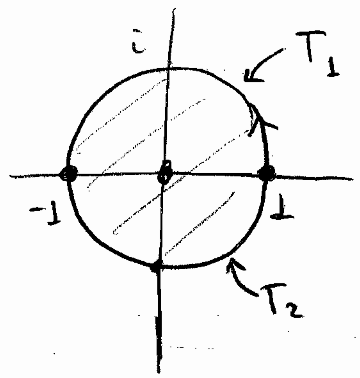
- Since conformal maps preserve angles, any circle that is tangent to unit circle at -1 will be mapped to a horizontal line:



- Since $f(1)=0$, any circle that goes through both $z=1$ and $z=-1$ will be mapped to a straight line through the origin:

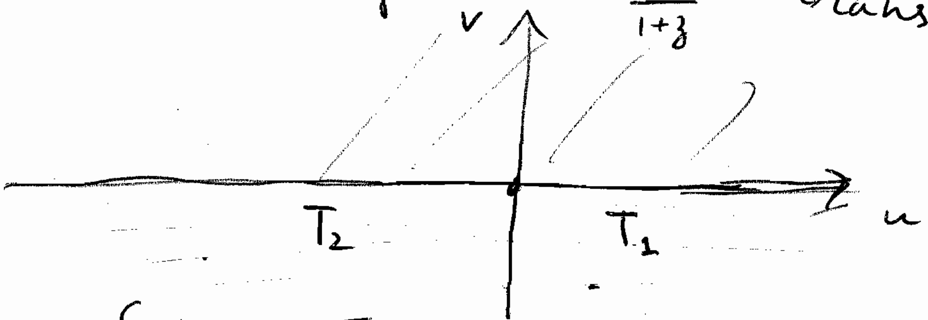


Ex: Find solution to



$$\begin{cases} \Delta T = 0 & \text{inside } C = \{|z|< \\ T = T_1 & \text{on upper boundary } z = e^{i\theta},\ \theta \in (0, \\ T = T_2 & \text{on lower boundary } z = e^{i\theta},\ \theta \in (\pi, \end{cases}$$

Soln: The map $\omega = \frac{i(1-z)}{1+z}$ transform this problem to:

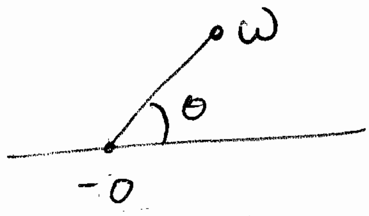


$$\begin{cases} T_{uu} + T_{vv} = 0, & v > 0 \\ T(u,0) = \begin{cases} T_1 & \text{if } u > 0 \\ T_2 & \text{if } u < 0 . \end{cases} \end{cases}$$

The solution to transformed problem is:

$$T = A + B\theta \quad \text{where} \quad \theta = \arg(\omega)$$

and A, B are chosen to satisfy the boundary conditions:



$$T_1 = A\ ; \quad T_2 = A + B\pi$$

$$\Rightarrow T = T_1 + \left(\frac{T_2 - T_1}{\pi}\right) \arctan\left(\frac{v}{u}\right),$$

Where the branch of arctan is chosen so that arctan $\in (0, \pi)$.

Now $u + iv = i \dfrac{(1-z)(1+\bar{z})}{|1+z|^2} =$

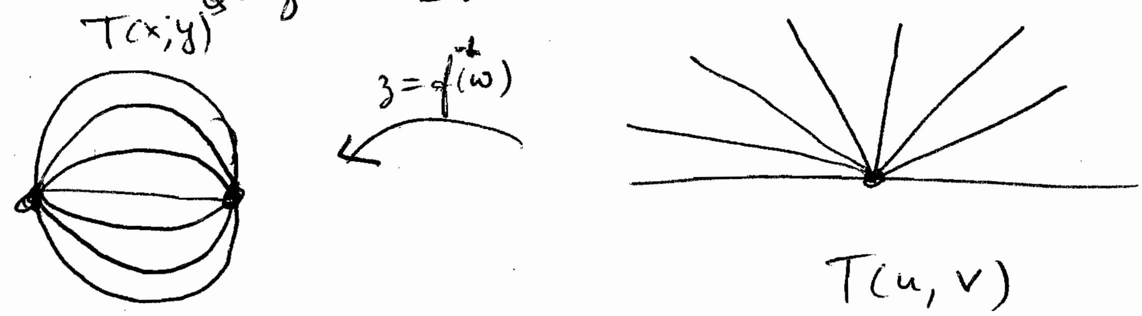
$$= \frac{i(1-x-iy)(1+x-iy)}{(x+1)^2+y^2}$$

$$= \frac{1}{(x+1)^2+y^2}\left((1+x)y + (1-x)y + i\left[(1+x)(1-x) - y^2\right] \right)$$

$$= \frac{1}{(x+1)^2+y^2}\left(2y + i(1-x^2-y^2)\right)$$

$$\Rightarrow \boxed{T = T_1 + \frac{T_2 - T_1}{\pi} \arctan\left(\frac{1-x^2-y^2}{2y}\right)}$$

To visualize the solution, note that the contour lines of T in the (u, v) plane are $\theta \equiv$ const, i.e. rays going through 0. Since these are mapped to circles, the contour lines of T in the (x,y) plane are circles through ± 1:

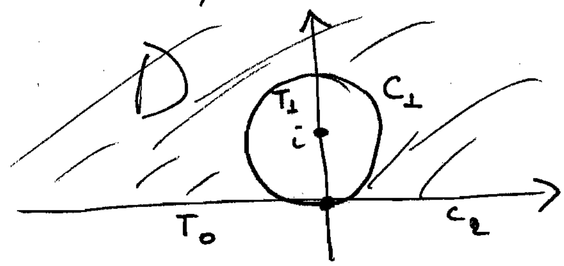


Ex

Let $C_1 = \{ z : |z-i| = 1 \}$

and let $C_2 = \{ z : \operatorname{Im} z = 0 \} = x$-axis.

Let $D = \{ z :$ above C_2, outside $C_1 \}$



Solve $\Delta T = 0$ inside D

with $\begin{cases} T = T_0 & \text{on } C_2 \\ T = T_1 & \text{on } C_1 \end{cases}$

Sol'n: We will use a bilinear map to map C_1 and C_2 into two straight lines.

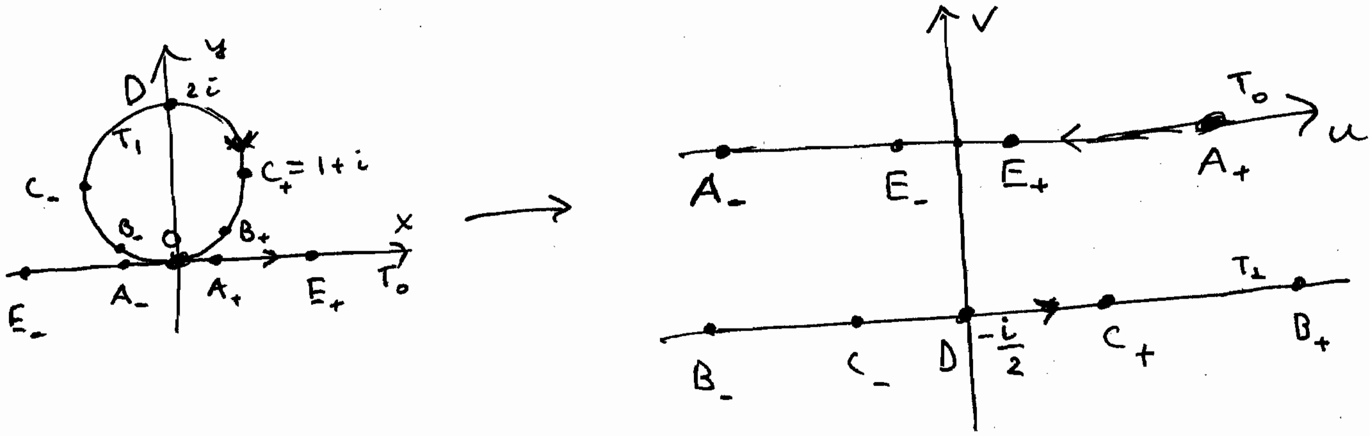
Consider $w = f(z) = \dfrac{az+b}{z+d}$. Then any circle/line going through $z=-d$ gets mapped into a line. So choose $d=0$; then C_1, C_2 become lines under D. So $f(z) = a + \dfrac{b}{z}$.

So choose $a=0$, $b=1$ [other choices for a, b will only translate/scale/rotate]

$$\Rightarrow \boxed{w = f(z) = \frac{1}{z}}, \qquad \begin{array}{l} w = u+iv \\ z = x+iy \end{array}$$

- $f(C_2) = \{u\text{-axis}\}$ • $f(0) = \infty$
- $f(2i) = -\dfrac{1}{2i}$, $f(1+i) = \dfrac{-i}{2} + \dfrac{1}{2}$, $f(-1+i) = \dfrac{-i}{2} - \dfrac{1}{2}$

$$\Rightarrow f(C_1) = \{ w = u+iv : v = -\tfrac{1}{2} \}$$



So $E = f(D) = \{ w : -\frac{1}{2} < \operatorname{Im} w < 0 \}$

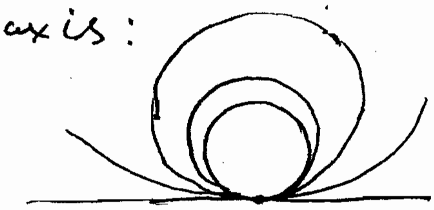
and we solve:
$$\begin{cases} T_{uu} + T_{vv} = 0 & \text{inside } E \\ T = T_0 & \text{if } v = 0 \\ T = T_1 & \text{if } v = -\frac{1}{2} \end{cases}$$

Such solution is: $\boxed{T = T_0 + 2(T_0 - T_1)\, v}$

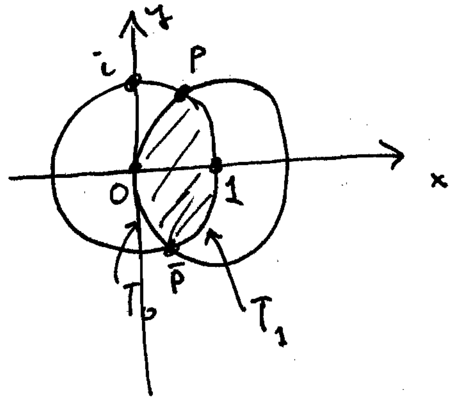
Now
$$\omega = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$
$$\Rightarrow \quad v = \frac{-y}{x^2+y^2}$$

$$\boxed{T = T_0 + 2(T_1 - T_0)\,\frac{y}{x^2+y^2}}$$

Contours of T are circles through 0, tangent to x-axis:



Pbm: Solve $\Delta T = 0$ on:



Sol: Let p be the point of intersection of the two circles. If $p = a + ib$ then

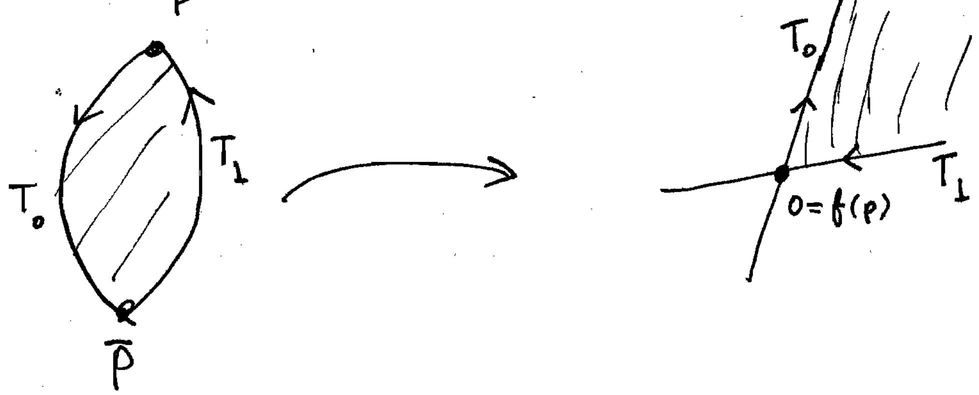
$$a^2 + b^2 = 1, \quad (a-1)^2 + b^2 = 1$$

$$\Rightarrow a^2 = (a-1)^2 \Rightarrow \boxed{a = \frac{1}{2}, \quad b = \frac{\sqrt{3}}{2}}$$

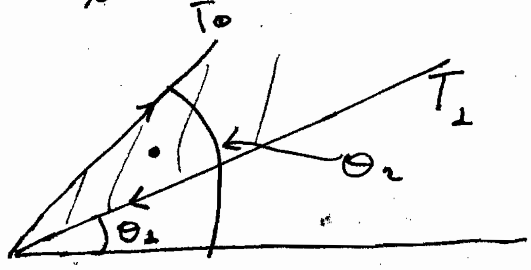
Now consider the map

$$w = f(z) = \frac{z - p}{z - \bar{p}}.$$

Since $f(z)$ has a pole at $\bar{p}$ where the two circles also intersect, their image are two straight lines, that intersect at $f(p) = 0$.



Next we solve $T_{uu}+T_{vv}=0$ inside a wedge:



Given the wedge angles θ_1, θ_2, the solution is given by $T = A + B\theta$

where $\theta = \arg(\omega)$ and $\begin{cases} T_1 = A + B\theta_1 \\ T_0 = A + B\theta_2 \end{cases}$.

It remains to determine the angles θ_1, θ_2.

Remark that for any analytic map,

if $\theta = \arg(z_1 - z_0)$ and $\omega_0 = f(z_0)$, $\omega_1 = f(z_1)$

then $\arg(\omega_1 - \omega_0) = \theta + \arg(f'(z_0))$ as $z_1 \to z_0$

<u>Proof</u>: Write $z_1 = z_0 + \rho e^{i\theta}$ and expand $f(z)$ about z_0:

$$f(z_1) = a_0 + (z_1 - z_0)a_1 + O(z_1 - z_0)^2 \ldots$$
$$= a_0 + a_1\rho e^{i\theta} + \ldots$$

Now $a_0 = f(z_0)$ and $\arg(\omega_1 - \omega_0) =$

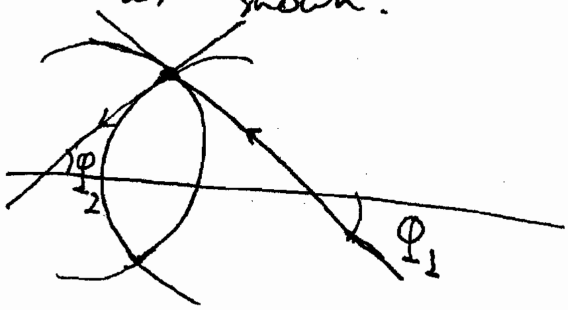
$$= \arg(a_1 \rho e^{i\theta}) = \arg a_1 + \theta$$
$$= \arg f'(z_0) + \theta .$$

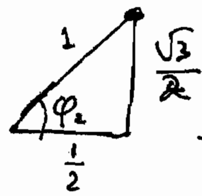
∎

Now $f'(p) = \frac{1}{p-\bar{p}} = \frac{1}{2bi} = -i\,\frac{1}{2b}$

$\Rightarrow$ $\boxed{\arg f'(p) = \pi.}$

So $\theta_1 = \pi + \varphi_1$, $\theta_2 = \pi + \varphi_2$ where φ_i are as shown:





$\rightarrow \varphi_2 = 60°$

$\Rightarrow$ $\varphi_2 = \frac{\pi}{3}$, $\varphi_1 = -\frac{\pi}{3}$;

$$\boxed{\theta_1 = \frac{2}{3}\pi, \quad \theta_2 = \frac{4}{3}\pi}$$

So $T_1 = A + B\frac{2}{3}\pi$, $T_0 = A + B\frac{4}{3}\pi$

$\Rightarrow$ $B = (T_0 - T_1)\frac{3}{2\pi}$; $A = 2T_1 - T_0$

$$\Rightarrow \boxed{T = (2T_1 - T_0) + \theta\,(T_0 - T_1)\frac{3}{2\pi}}$$

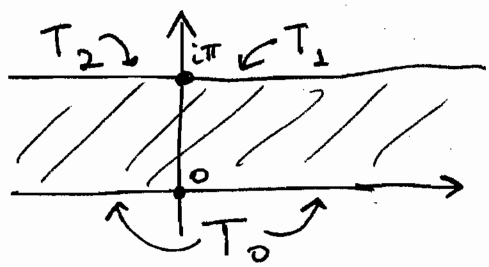
where $\theta = \arg \omega = \arg \frac{3-p}{3-\bar{p}}$

$$\Rightarrow \boxed{\theta = \arctan\left(\frac{y-b}{x-a}\right) - \arctan\left(\frac{y+b}{x-a}\right)}$$

$\boxed{\text{where } a = \frac{1}{2}, \quad b = \frac{\sqrt{3}}{2}}$

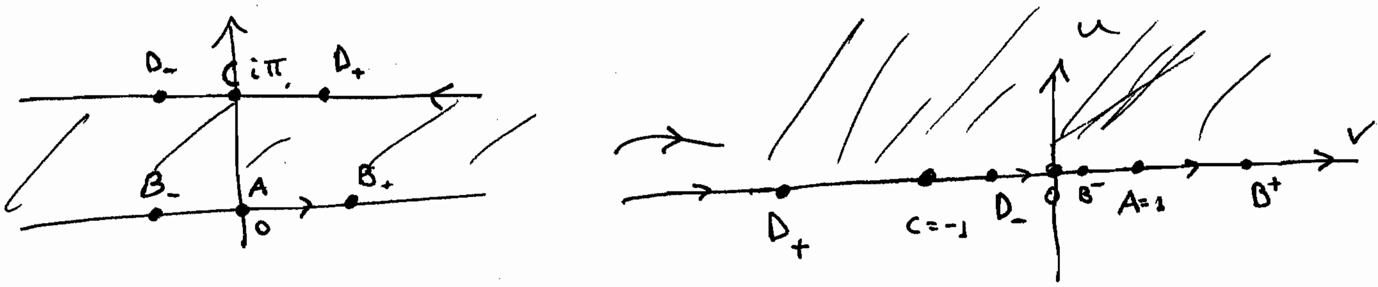
<u>Ex</u>: Solve $\Delta T = 0$ on $D = \{ z = x + iy : 0 < y < \pi \}$

with B.C:

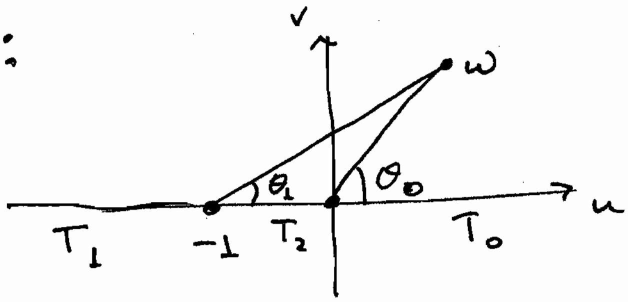


<u>Sol</u>: The map $\omega = e^{z}$ maps D into $\{ \omega : \operatorname{Im} \omega > 0 \}$:

$$e^{z} = e^{x} e^{iy} \quad ; \quad y \in [0, \pi] , \quad x \in \mathbb{R}$$



The B.C. become:



~~Given~~ So let $T = A + B\theta_0 + C\theta_1$

Where $\theta_0 = \arg \omega$; $\theta_1 = \arg(\omega - (-1))$

Then T is ~~anal~~ harmonic and choose A, B, C to match B.C.:

$\omega = T_0 \Rightarrow \theta_0, \theta_1 = 0 \Rightarrow A = T_0$;

$\omega = T_2 \Rightarrow \theta_0 = \pi, \ \theta_1 = 0$; $T_2 = T_0 + B\pi$ $\quad B = \dfrac{T_2 - T_0}{\pi}$

$\omega = T_1 \Rightarrow \theta_0 = \pi, \theta_1 = \pi$; $T_1 = T_2 + C\pi$ $\quad C = (T_1 - T_2)/\pi$

$$\Rightarrow \quad T = T_0 + \frac{T_2 - T_0}{\pi}\,\theta_0 + \frac{T_2 - T_1}{\pi}\,\theta_1$$

where $\quad \theta_0 = \arctan\left(\frac{v}{u}\right); \quad \theta_1 = \arctan\left(\frac{v}{u+1}\right)$

where $\quad u + iv = w = e^{x+iy}$

$$\Rightarrow \quad u = e^x \cos y, \quad v = e^x \sin y$$

$$\Rightarrow \boxed{T = T_0 + \frac{T_2 - T_0}{\pi}\, y + \frac{T_2 - T_1}{\pi} \arctan\left(\frac{\sin y}{\cos y + e^{-x}}\right)}$$

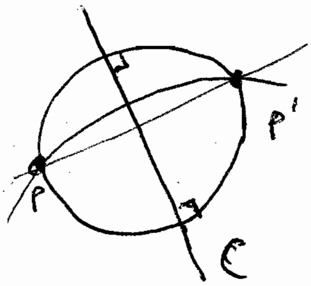
Check: if $T_2 = T_1 \Rightarrow T$ is linear in y as expected.

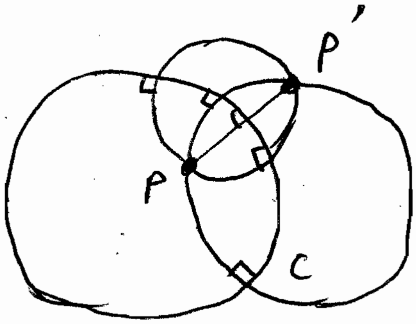
Example: Plot contour lines of T above with $T_0 = 0$, $T_1 = 1$, $T_2 = -1$.

Sol'n: See the attached Maple worksheet.

Bilinear maps and symmetry points

Given a line or a circle C, two points p and p' are said to be <u>symmetric w.r.t. C</u> if any circle or line going through p and p' intersects C orthogonally.





Given a point p and a circle or line C, the point p' is a <u>reflection of p in C</u> if p and p' are symmetric w.r.t. C.

Note that if C is a line, then the above definition coincides with the usual mirror reflection.

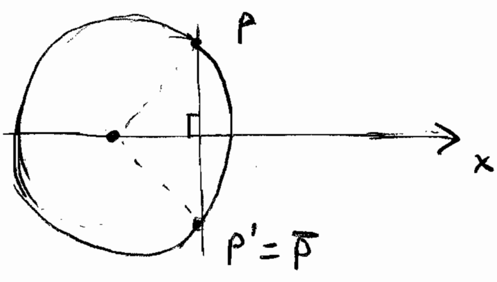
<u>Claim</u>: Given a circle C; a bilinear map
$w = f(z) = \frac{az+b}{cz+d}$ and p, p' that are symmetric with respect to C, then the images of p, p', $w = f(p)$, $w' = f(p')$ are symmetric with respect to the image $D = f(C)$ of C.

Proof: Recall that Bilinear maps map circles and lines into circles and lines, so D is a circle or a line. Now given a circle or a line E through $\omega = f(p)$ and $\omega' = f(p')$, its pre-image $F = f^{-1}(E)$ is a circle or line through p and p'. Since p and p' are symmetric w.r.t. C, F is orthogonal to C. But $f(z)$ is analytic so it preserves angles. It follows that E is orthogonal to D. ∎

Remark: if p is the center of a circle C, then its reflection is $p' = \infty$, since any circle through p and ∞ is a straight line through p and thus intersects C orthogonally. Conversely, if $p = \infty$ then its reflection in C is the center of C.

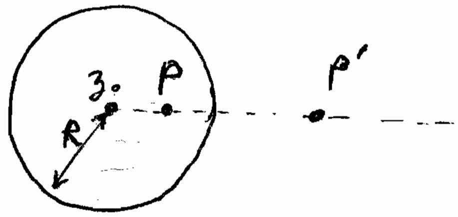
Claim: If C is the x-axis then given any point p, its reflection is given by $p' = \bar{p}$.

Proof: any circle through p and p' has its centre on the x-axis and thus intersects the x-axis orthogonally.



Thm: Given a circle C with center z_0 and radius R, and given a point p, its reflection p' lies on the ray through p and z_0 and its distance from z_0 is determined by

$$|p - z_0|\,|p' - z_0| = R^2.$$



Proof: Consider a special case, $R=1$, $z_0=0$, $C=$ unit circle.

Recall that the map $w = f(z) = i\,\dfrac{(1-z)}{1+z}$ maps the unit circle to x-axis. If p, p' are symmetric w.r.t. unit circle, then $f(p)$ and $f(p')$ are symmetric w.r.t. x-axis, that is:

$$i\,\frac{(1-p)}{1+p} = \overline{\left(i\,\frac{(1-p')}{1+p'}\right)} = -i\left(\frac{1-\overline{p'}}{1+\overline{p'}}\right)$$

Solving for $\overline{p}'$ we get:

$$(1+\overline{p}')(1-p) = (\overline{p}'-1)(1+($$

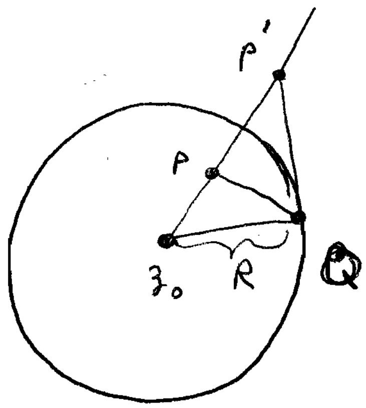
$$\overline{p}' = \frac{1+p+1-p}{1+p-(1-p)} = \frac{1}{p}$$

$$\Rightarrow \boxed{p' = \frac{1}{\overline{p}}}\ ; \text{ that is } p' = p\,\frac{1}{|p|^2}$$

Hence p and p' are along the same ray from 0 and $|p|\,|p'| = 1$. So the theorem is proved for the unit circle. The more general case is obtained by considering scalings and translations ∎

<u>Remark</u>: Reflection inside a circle has the following geometric interpretation:

If P and P' are reflections of each other in C and Q is a point on the boundary of C then the ratio $\dfrac{|P-Q|}{|P'-Q|}$ is constant, independent of Q:



<u>Proof</u>: The triangles

$$\triangle z_0 Q P \sim \triangle z_0 P' Q$$

since they both share angle z_0 and since the ratio $\dfrac{|Q-z_0|}{|P'-z_0|} = \dfrac{|P-z_0|}{|Q-z_0|}$

[P', P are symmetric $\Leftrightarrow$ $|P-z_0|\,|P'-z_0| = R^2$]

Thus $$\frac{|P-Q|}{|P'-Q|} = \frac{|z_0-Q|}{|z_0-P'|} = \frac{|z-P|}{|z_0-Q|} = \frac{R}{|z-P'|} = \frac{|z-P|}{R}$$

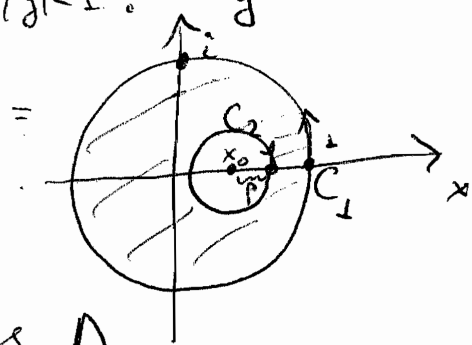
is idependent of Q. ∎

Symmetric points have the following useful interpretation: if p and p' are symmetric w.r.t. C then the map $w = f(z) = \dfrac{z-p}{z-p'}$ maps C into a circle $D = f(C)$ which is centered at the origin.

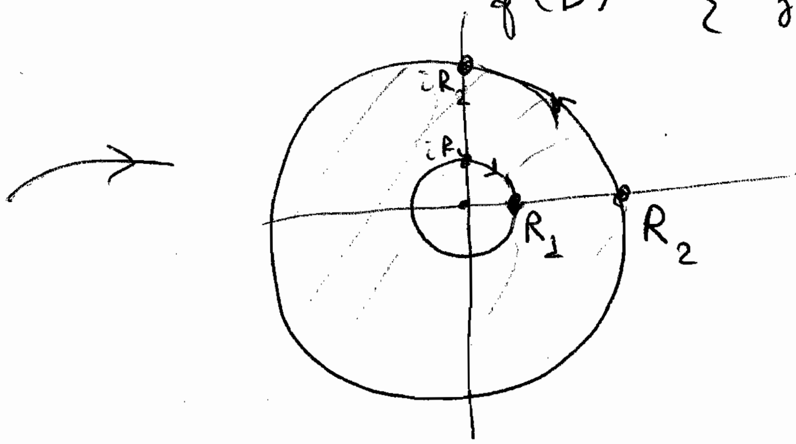
Proof: $f(z)$ maps p and p' into 0 and ∞ which become symmetry points of D, hence 0 is the center of D. ■

Application: Consider a domain D consisting of the inside of the unit circle C_1 minus a smaller circle C_2 entirely inside $|z|<1$:

$$D = \{z : |z| < 1 \text{ and } |z - x_0| > \rho\}$$

$D =$ 

Find a bilinear map that maps D into an annulus $f(D) = \{z : R_1 < |z| < R_2\}$



Sol'n: The desired map is $f(z) = \frac{z-p}{z-p'}$,

where p, p' are symmetric w.r.t. both C_1 and C_2.

By symmetry, write $p = \alpha_1 \in \mathbb{R}$ and $p' = \alpha_2 \in \mathbb{R}$;

then we have: $\alpha_1 \alpha_2 = 1$ and $(\alpha_1 - x_0)(\alpha_2 - x_0) = \rho^2$

So $\alpha_1 = \alpha$; $\alpha_2 = \alpha^{-1}$ where

$$\alpha = \frac{x_0^2 + 1 - \rho^2 \pm \sqrt{(x_0^2+1-\rho^2)^2 - 4x_0^2}}{2x_0}.$$

The two radii are then $R_1 = |f(1)|$

$R_2 = |f(x_0 \pm \rho)|$

and $f(z) = \frac{z-\alpha}{z-\frac{1}{\alpha}} \Rightarrow R_1 = \left|\frac{1-\alpha}{1-\frac{1}{\alpha}}\right| = |\alpha|$.

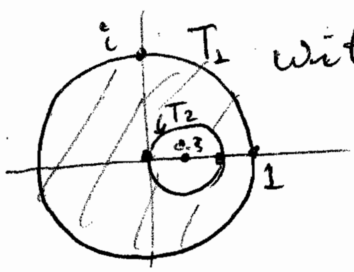
Pbm: Let $C_1 = \{|z|=1\}$ and let

$C_2 = \{z : |z-0.3| = 0.3\}$

Let $D = \{z : z \text{ inside } C_1 \text{ and outside } C_2\}$.

Solve $\Delta T = 0$ inside D

with $T = T_1$ on C_1, $T = T_2$ on C_2.



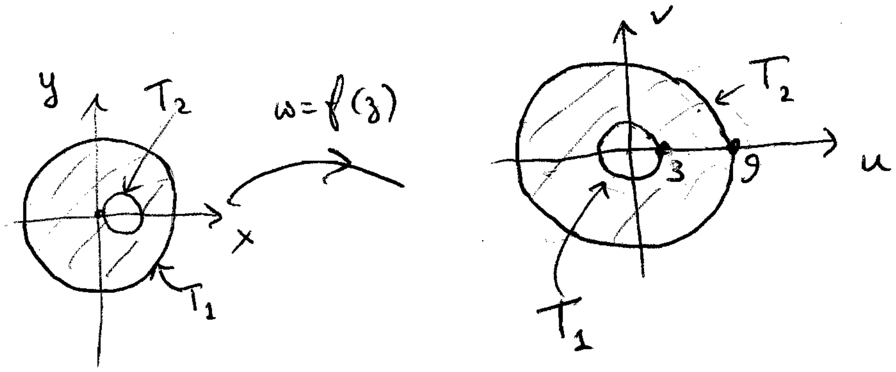
Sol'n: Applying previous example with $x_0 = 0.3$, $\rho = 0.3$

we get $$\alpha = \frac{1 \pm \sqrt{1 - 0.09 \cdot 4}}{0.6} = \frac{1 \pm 0.8}{0.6} = \frac{1}{3} \text{ or } 3$$

Take $\alpha = 3$, $w = f(z) = \dfrac{z-3}{z-\frac{1}{3}}$.

Then $f(1) = 3 = R_1$; $f(x_0 - \rho) = 9 = R_2$

So in the (u, v) plane we get:



Now the ~~transformed~~ problem has a radial symmetry and its solution is given by

$$T = A \ln |w| + B \quad \text{where}$$

A, B are chosen to match the boundary conditions.

That is, $$\left.\begin{aligned} T_1 &= A \ln 3 + B \\ T_2 &= A \ln 9 + B \end{aligned}\right\} \Rightarrow A = \frac{T_2 - T_1}{\ln 9 - \ln 3} = \frac{T_2 - T_1}{\ln 3}$$

$$B = T_1 - (T_2 - T_1)\frac{\ln 3}{\ln 3} = 2T_1 - T_2$$

$$\Rightarrow \boxed{T = \frac{T_2 - T_1}{\ln 3} \ln \left| \frac{z-3}{z-\frac{1}{3}} \right| + 2T_1 - T_2}$$