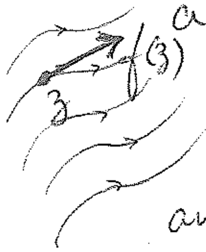
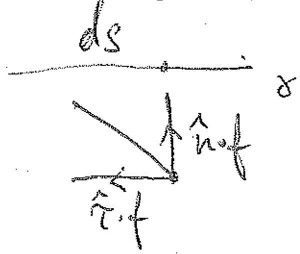
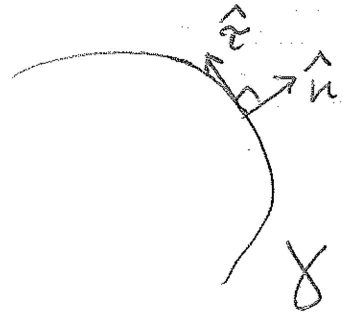
Source: Fisher, Complex Variables.

<u>Flow</u>: Imagine a fluid inside some domain $D \subset \mathbb{C}$. At each point $z \in D$, let $f(z) \in \mathbb{C}$ be the velocity of the fluid there. This function is a <u>flow</u> which defines a <u>vector field</u>.



Given a curve $\gamma(t) \in \mathbb{C}$, the total amount of fluid that crosses γ in unit time is given by $\int_\gamma f \cdot \hat{n}\, ds$





and the amount of fluid tangential to γ is given by $\int_\gamma f \cdot \hat{\tau}\, ds$.

Now denote $f(z) = u(z) + i v(z)$; $z = x + iy$

~~and so~~ and $\gamma = (x(t), y(t))$, $t \in [a, b]$;

then $\vec{\tau} = (x'(t), y'(t))$;

$$\hat{\tau} = \frac{(x', y')}{\sqrt{x'^2 + y'^2}} \quad ; \quad ds = \sqrt{x'^2 + y'^2}\, dt$$

and $\vec{n} = (y', -x')$; $\hat{n} = \dfrac{(y', -x')}{\sqrt{x'^2 + y'^2}}$

so that $\quad f\cdot\hat{n}\,ds = (u,v)\cdot\dfrac{(y',-x')}{\sqrt{x'^2+y'^2}}\sqrt{x'^2+y'^2}\,dt$

$$= (u,v)\cdot(y',-x')\,dt$$

$$= \mathrm{Im}(\bar{f}\,dz)$$

Thus $\quad \displaystyle\int_\gamma f\cdot\hat{n}\,ds = \mathrm{Im}\left(\int_\gamma \bar{f}(z)\,dz\right)$

and similarly, $\quad \displaystyle\int_\gamma f\cdot\hat{\tau}\,ds = \mathrm{Re}\left(\int_\gamma \bar{f}\,dz\right).$

The integral $\displaystyle\int_\gamma f\cdot\hat{n}\,ds$ is called the <u>flux</u> through γ;

similarly, $\displaystyle\int_\gamma f\cdot\hat{\tau}\,ds$ is called the <u>circulation</u> through γ.

- Alternatively, if f represents force field, then $\displaystyle\int_\gamma f\cdot\hat{n}\,ds$ is the <u>work</u> done along curve γ.

<u>Def</u>: the flow is <u>sourceless</u> or <u>fluxless</u> inside some domain D if $\displaystyle\int_\gamma f\cdot\hat{n}\,ds$ for any closed curve γ in D.

Similarly, $f(z)$ is <u>irrotational</u> if $\displaystyle\int_\gamma f\cdot\hat{\tau}\,ds = 0$ $\forall$ closed $\gamma\subset D$.

<u>Green's thm</u> says that

$$\int_\gamma u\,dx + v\,dy = \int_\Omega (v_x - u_y)\,dx\,dy$$

where Ω is the inside of a closed curve γ, traversed counter-clockwise.

Note also that $u\,dx + v\,dy = f\cdot\hat{\tau}\,ds$

So if the flow is <u>irrotational in D</u> then

$$\int_\Omega (v_x - u_y)\,dx\,dy = 0 \quad \forall\, \Omega \subset D$$

$$\Rightarrow \boxed{v_x = u_y \quad \forall\, (x,y) \in D}$$

Similarly,
$$\int_\gamma f\cdot\hat{n}\,ds = \int_\gamma u\,dy - v\,dx$$
$$= \int_\Omega (u_x + v_y)\,dx\,dy$$

So if the flow is <u>sourceless in D</u> then
$$u_x = -v_y \quad \forall\, (x,y) \in D$$

Now suppose $\overline{f(z)}$ is <u>analytic</u>, $\overline{f} = u - iv$.
Then C-R eq'ns are:

$$u_y = v_x, \quad u_x = -v_y.$$

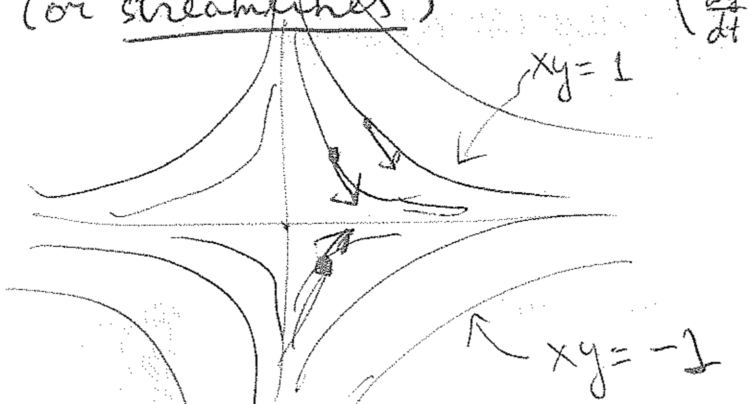
That is if $f(z)$ is <u>both</u> sourceless and irrotational then $(u, -v)$ satisfy C-R $\Rightarrow$ <u>$\overline{f}$ is analytic</u>

i.e. $\int_\gamma \overline{f}\, dz = 0$ $\forall$ closed curve $\gamma \subset D$

Conversely, if $\overline{f}$ is analytic then f is <u>locally</u> sourceless and irrotational, i.e. $\forall\, z \in D$, $\exists\, \Omega \subset D$ with $z \in \Omega$ s.t. f is irrot. & sourceless inside the smaller domain Ω.

<u>Ex 1</u>: $f(z) = \overline{z} = (x, -y)$. Then $\overline{f} = z$ is <u>analytic</u> so that f is both sourceless & irrotational.

<u>Flow curves</u>: set $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \Rightarrow \frac{dx}{dy} = -\frac{x}{y}$
(or <u>streamlines</u>)



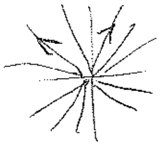
$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\ln x = c - \ln y$$

$$\Rightarrow \boxed{xy = \text{const}}$$

Ex2: $f(z) = (x,y) = z$. Then $u = x,\ v = y$

Then $$\int_\gamma f\cdot\hat{n}\,ds = \int_\gamma u\,dy - v\,dx = \int_\Omega \underbrace{\left(u_x - (-v)_y\right)}_{2}\,dx\,dy$$
$$= 2\,\text{area}(\Omega)$$



$\Rightarrow$ f is <u>non-conservative</u>

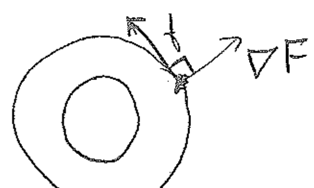
and $$\int_\gamma f\cdot\hat{\tau}\,ds = \int u\,dx + v\,dy = \int \underbrace{\left(\overset{y}{v_x} - \overset{x}{u_y}\right)}_{0}\,dx\,dy$$

$\Rightarrow$ f is <u>irrotational</u>

Ex3: $f(z) = (y,-x)$ Streamlines: $x^2+y^2 = C$

[check: $F(x,y) = x^2+y^2$; $\nabla F\cdot f = (2x,2y)\cdot(y,-x) = 0$

so that



]

But $$\int_\gamma f\cdot\hat{\tau} = 2\,\text{area}(\Omega)\ ; \quad \int f\cdot\hat{n} = 0$$

$\Rightarrow$ f is conservative but <u>not</u> irrotational.

Def: A flow f is <u>ideal</u> if it is locally sourceless & irrotational

$\Leftrightarrow$ $\bar{f}$ is harmonic.

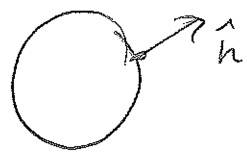
Ex: $f = \frac{1}{z}$ is an ideal flow; locally irrot. cons.

But not globally:

$$f \simeq \frac{z}{|z|^2} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

• If $\gamma = (r\cos\theta, r\sin\theta) = re^{i\theta}$, $\theta = 0 \ldots 2\pi$

then $\hat{n} = (\cos\theta, \sin\theta) = e^{i\theta}$ $\quad ds = r\,d\theta$



and $$\int_\gamma f \cdot \hat{n}\, ds = \cancel{\frac{1}{2\pi}} \int_0^{2\pi} \left(\frac{r\cos\theta}{r^2}, \frac{r\sin\theta}{r^2} \right) \cdot (\cos\theta, \sin\theta)\, r\, d\theta$$

$$= 2\pi \neq 0$$

[this is due to singularity at 0]

<u>Streamlines:</u> given an <u>ideal</u> flow $f(z)$, let $G(z)$ be s.t. $G'(z) = \underbrace{\overline{f}(z)}_{\text{analytic}}$

<u>Claim:</u> streamlines of f are given by

$$\operatorname{Im} G = \text{const}.$$

<u>Pf:</u> write $G = \mathcal{U} + i\mathcal{V}$; $f = u + iv$;

$$\partial_x G = \frac{\partial}{\partial z} G \underbrace{\frac{\partial z}{\partial x}}_{1} = u - iv \Rightarrow \mathcal{U}_x + i\mathcal{V}_x = u - iv$$

and

$$\partial_y G = \frac{\partial}{\partial z} G \underbrace{\frac{\partial z}{\partial y}}_{i} \Rightarrow \left(\mathcal{U}_y + i\mathcal{V}_y\right) = (u - iv)i$$
$$= v + iu$$

$$\Rightarrow \boxed{V_x = -v, \quad V_y = u}$$

But then $\nabla V(x,y) \cdot f = (-v, u) \cdot (u, v) = 0$

$\Rightarrow$ $V = \text{const.}$ are tangent to f

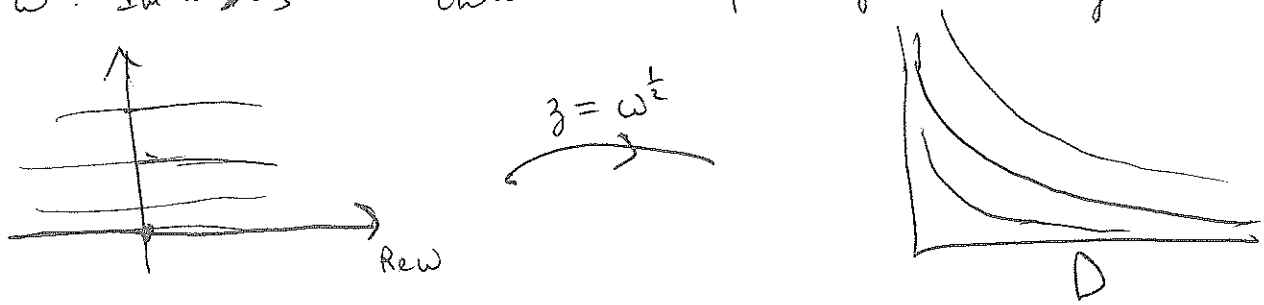
<u>Def:</u> G is called the <u>potential</u> of f and V is called the <u>stream function</u>.

<u>Ex:</u> $f = \overline{z}$; $\overline{f} = z$; $G = \int z \, dz = \frac{z^2}{2}$

$$G = \underbrace{\frac{x^2 - y^2}{2}}_{\mathcal{U}} + \underbrace{\frac{2ixy}{2}}_{i\mathcal{V}} \Rightarrow \boxed{V = xy}$$

$xy = \text{const}$ are streamlines.

Ex: Consider a map $H(\omega) = \omega^{\frac{1}{2}}$ maps upper plane $\{\omega : \mathrm{Im}\,\omega > 0\}$ into a first quadrant $z = \omega^{\frac{1}{2}}$:



It also maps the streamlines $\mathrm{Im}(\omega) \equiv \text{const}$
or $\omega = t + ic$, $t \in \mathbb{R}$

into streamlines $z = (t + ic)^{\frac{1}{2}}$ in D

The <u>potential</u> of the flow in D is the <u>inverse</u> of $H(\omega)$: $z = \omega^{\frac{1}{2}} \Leftrightarrow \omega = z^2 = G(z)$

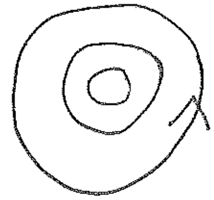
Indeed $\mathrm{Im}\, G = 2xy \equiv \text{const}$ ~~is the~~ are the streamlines.

Ex: $f = \frac{i}{z}$; $\overline{f} = \frac{i}{z} \Rightarrow G = \int \overline{f} = i \ln z$

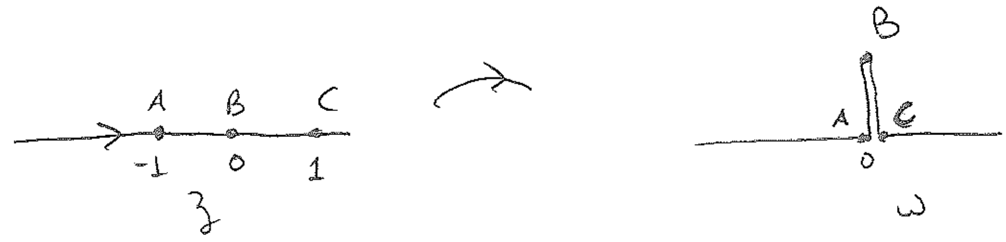
$$G = i\left(\ln|z| + \arg(z)\, i\right) = -\arg z + i \underbrace{\ln|z|}_{V}$$

$\Rightarrow V = \frac{1}{2}\ln|x^2+y^2| = \text{const}$ are streamlines

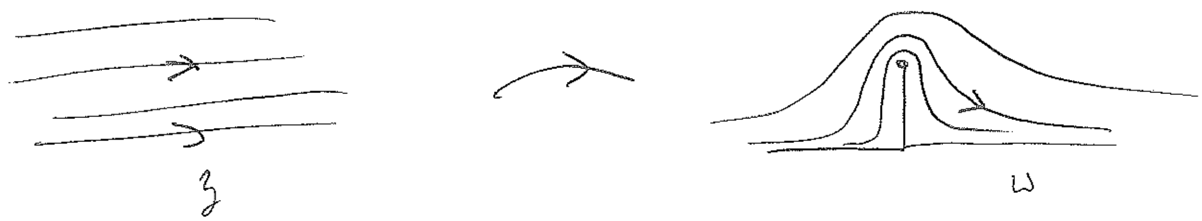
or $x^2+y^2 = \text{const}.$



<u>Ex:</u> Let $\quad w = (z^2-1)^{\frac{1}{2}}$. It maps the real axis into:



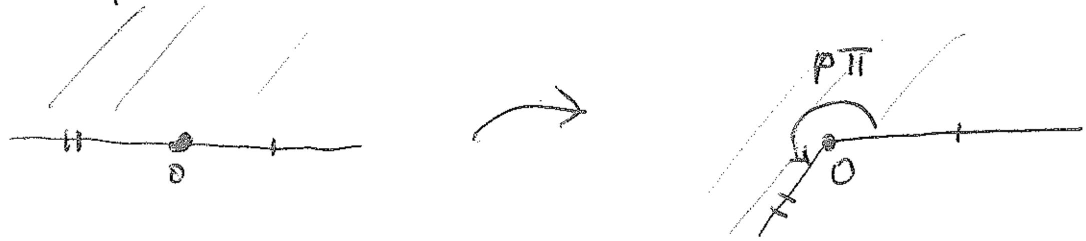
and maps the streamlines $z = t + ic$ into:



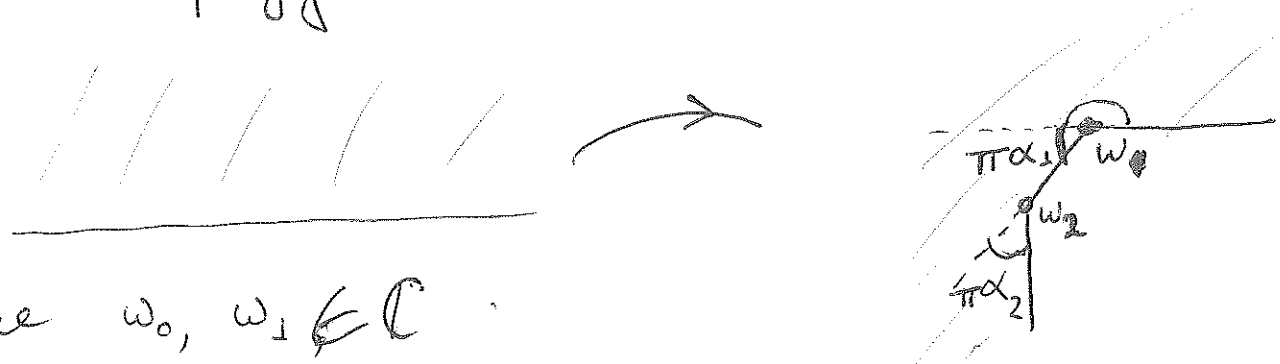
The resulting streamlines represent an ideal flow over the obstacle.

Schwartz-Christoffel transform:

The map $w = z^p$ transforms the upper half-plane into a wedge of angle $p\pi$:



Q: Find a map that transforms $\mathcal{U} = \{(x,y),\ y>0\}$ into a polygon as shown:



[where $w_0, w_1 \in \mathbb{C}$
and $\alpha_1, \alpha_2 \in (-\pi, \pi)$.]

Answer: If $w = f(z)$ is such a map, then its derivative is given by

$$f'(z) = A\,(z - X_1)^{\alpha_1}(z - X_2)^{\alpha_2}$$

where $A \in \mathbb{C}$, and $X_1, X_2 \in \mathbb{R}$ are to be adjusted and where α_1, α_2 are the angles in question.

SC2

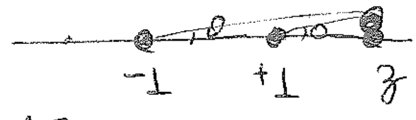
To see this, note that it is enough to map the boundaries; then the interior is automatically mapped.

Take for example $A = 1$, $x_1 = +1$, $x_2 = -1$:

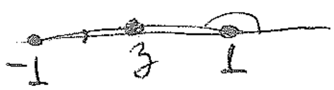
$$f'(z) = (z-1)^{\alpha_1}(z+1)^{\alpha_2}$$

Suppose $\mathrm{Im}(z) = 0$:

- If $z > 1$ then $\arg f' = 0$



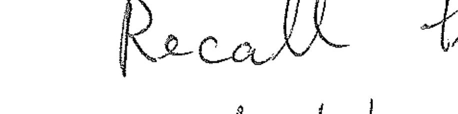
- If $z \in (-1, 1)$ then $\arg\left[(z-1)^{\alpha_1}\right] = \pi\alpha_1$



and $\arg\left[(z+1)^{\alpha_2}\right] = 0$

So $\arg(f') = \pi\alpha_1$.

- If $z < -1$ then $\arg(z-1)^{\alpha_1} = \pi\alpha_1$



$\arg(z+1)^{\alpha_2} = \pi\alpha_2$

So $\arg(f') = \pi\alpha_1 + \pi\alpha_2$.

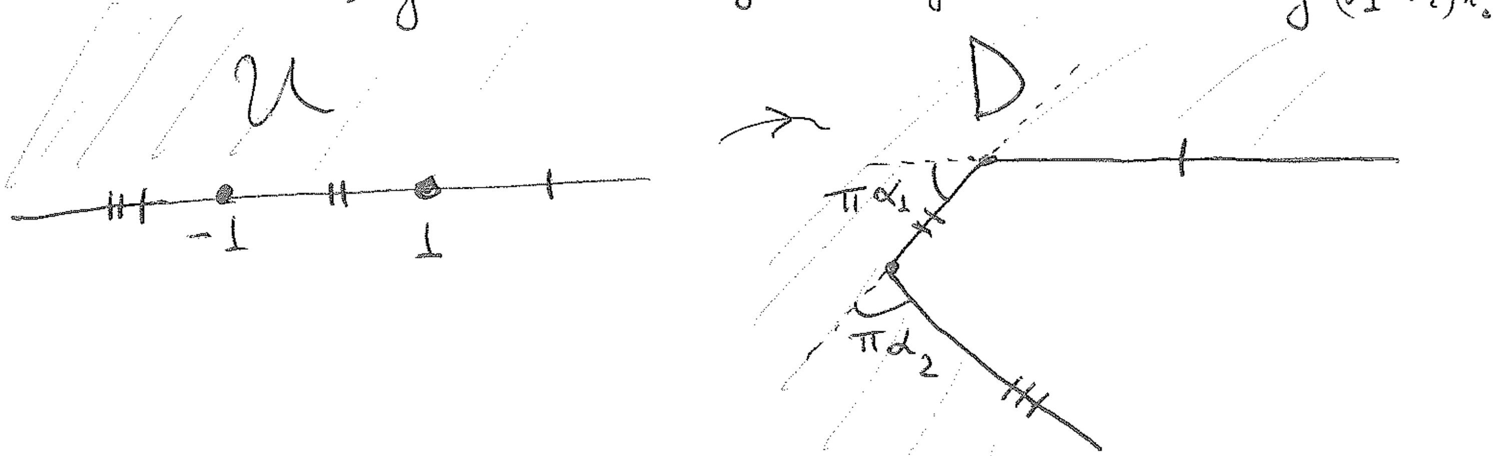
Recall that locally (for z near z_0) an analytic map stretches an image by $|f'(z_0)|$ and rotates it by $\arg f'(z_0)$.

[since locally, we can write $w = a_0 + a_1(z-z_0) + \ldots$ where $a_0 = f(z_0)$, $a_1 = f'(z_0)$; set $a_1 = e^{i\theta}R$; then $\theta = \arg f'(z_0)$ and $w = a_0 + Re^{i\theta}(z-z_0)$ rotates by angle θ]

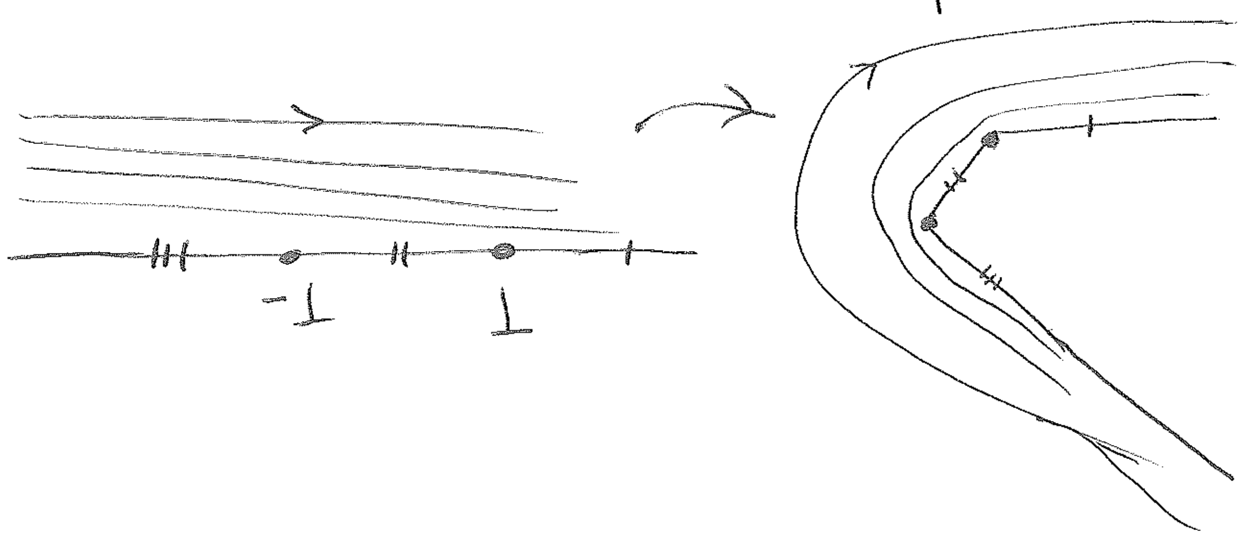
<u>So</u>: • The ray $z > 1$ gets mapped into a ray [$\arg f' = 0 \quad \forall z > 0$]

• Line segment $-1 < z < 1$ gets <u>rotated</u> by $\alpha_1 \pi$

• ~~Line segment~~ ray $z < -1$ gets rotated by $(\alpha_1 + \alpha_2)\pi$:

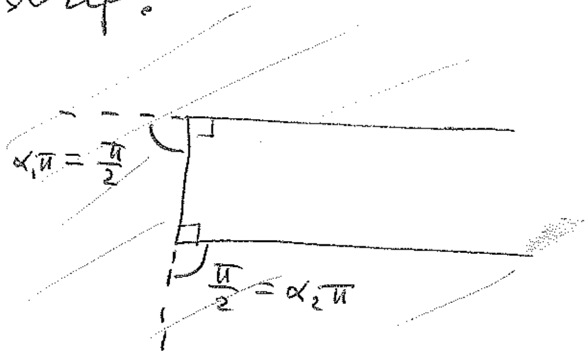


• Horizontal lines $y = c > 0$ get mapped into <u>streamlines</u> of D:



Ex 1: Flow across one-sided strip:

- $\alpha_1 = \alpha_2 = \frac{1}{2}$



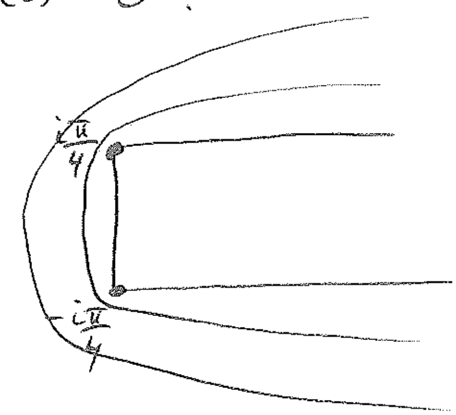
$$f' = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} = i(1-z^2)^{\frac{1}{2}}$$

This can be integrated:

$$\int(1-z^2)^{\frac{1}{2}}dz \underset{z=\sin t}{=} \int \cos^2 t = \int \frac{1+\cos 2t}{2} = \frac{t}{2} + \frac{\sin 2t}{4}$$
$$= \frac{t}{2} + \frac{\sin t \cos t}{2}$$
$$= \frac{\arcsin z}{2} + \frac{z(1-z^2)^{\frac{1}{2}}}{2}$$

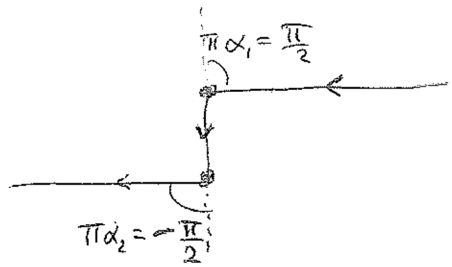
$$\Rightarrow \boxed{f(z) = \frac{i}{2}\left(z(z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} + \arcsin z\right)}$$

- $f(1) = i\frac{\pi}{4}$, $f(-1) = -\frac{i\pi}{4}$, $f(0) = 0$:



Ex 2 : Flow over a vertical step :

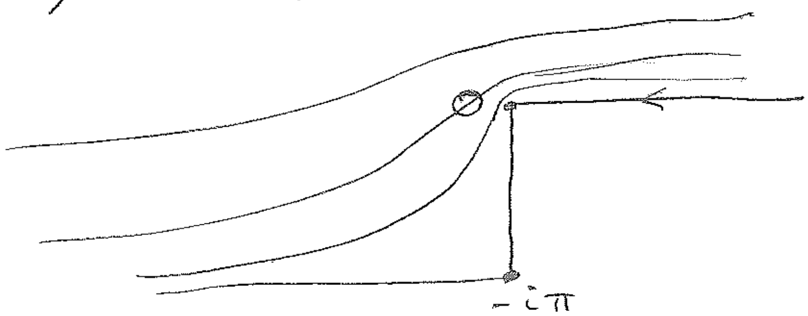
$\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$



$$f'(z) = (z-1)^{\frac{1}{2}}(z+1)^{-\frac{1}{2}}$$

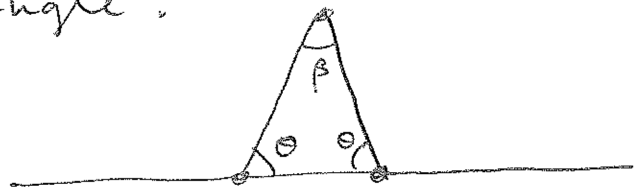
$$\Rightarrow \quad f(z) = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} - \ln\left(z + (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}\right)$$

$f(1) = 0$, $\quad f(-1) = -\ln(e^{i\pi}) = -i\pi$



Ex3: Flow past a triangle:

$2\theta + \beta = \pi$

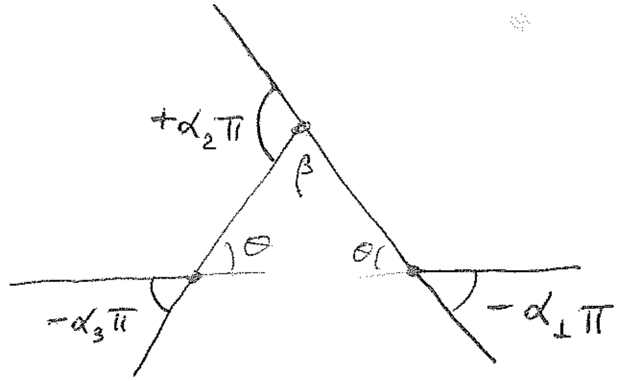


Take $f'(z) = (z-1)^{\alpha_1} z^{\alpha_2} (z+1)^{\alpha_3}$

where $\pi \alpha_1 = -\theta$

$\pi \alpha_3 = -\theta$

and $\pi \alpha_2 = +\pi - \beta$



$\Rightarrow \quad \alpha_1 = \alpha_3 = -\frac{\theta}{\pi}$; $\quad \alpha_2 = \frac{2\theta}{\pi}$

$$\Rightarrow \quad \boxed{f'(z) = (z-1)^{-\frac{\theta}{\pi}} (z+1)^{-\frac{\theta}{\pi}} z^{\frac{2\theta}{\pi}}}$$

Special case: Take $\theta = \frac{\pi}{2}$ [flow around vertical rod]

Then $f'(z) = (z-1)^{-\frac{1}{2}} (z+1)^{-\frac{1}{2}} z = (z^2-1)^{-\frac{1}{2}} z$

Then $\boxed{f(z) = (z^2-1)^{\frac{1}{2}}}$

- $f(0) = i$; $\quad f(\pm 1) = 0$

[tip of the rod is at $y = i$, $x = 0$]