1. (a) Expand the function $f(x) = e^{ax}$ using Periodic Fourier series on an interval $[-\pi, \pi]$. That is, write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(inx)$$

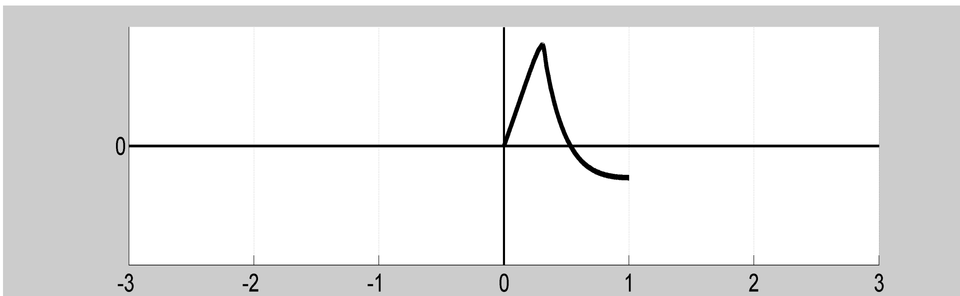
and compute the coefficients c_n.

(b) By reusing part (a), rewrite the Fourier series using sin/cos; that is write

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(inx) + b_n \sin(inx)$$

(c) Evaluate series in (b) at $x = 0$ and at $x = \pi$. What interesting identities does this give?

2. A function $f(x)$ on an interval from 0 to 1 looks like this:



(a) This function was expanded in a Fourier series of the form

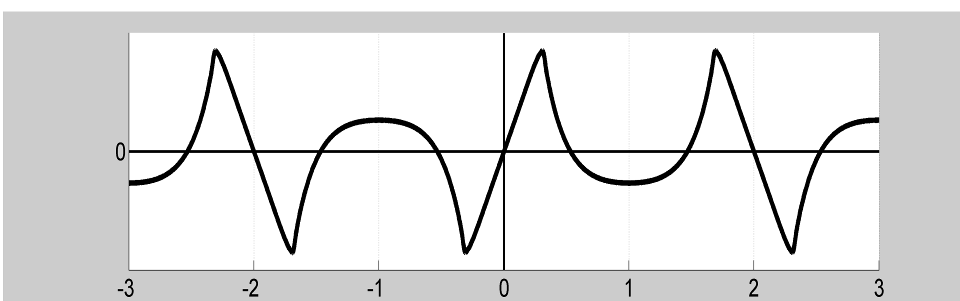
$$f(x) = \sum a_n \cos(n\pi x).$$

Sketch the graph of $f(x)$ when extended on the interval $[-3, 3]$ using this series.

(b) This function was expanded in a certain Fourier series of the form

$$f(x) = \sum b_n \phi_n(x).$$

and extended on an interval from -3 to +3, resulting in the following graph:



Determine $\phi_n(x)$.

3. Let $f(x) = x^2$. Consider four different representations of $f(x)$:

(1) $f_1(x) = \sum b_n \sin(nx), \;, \quad x \in (0, \pi);$

(2) $f_2(x) = \sum a_n \cos(nx), \quad x \in (0, \pi);$

(3) $f_3(x) = \sum a_n \cos(nx) + \sum b_n \sin(nx), \quad , \quad x \in (-\pi, \pi);$

(4) $f_4(x) = \sum b_n \sin\left(\left(n + \frac{1}{2}\right) x\right), \quad x \in (0, \pi).$

Without computing the coefficients, draw how $f_i(x), i = 1..4$ looks like when extended on the whole real line.

4. Solve

$$\begin{aligned} u_{tt} &= 2u_{xx} \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 3\sin(x) \\ u_t(x,0) &= 4\sin(2x) \end{aligned}$$

5. (Exercise 4.7.6) 4.7.6: Imagine that a stringed musical instrument falls on the floor. Suppose that the length of the string is 1 and $a = 1$. When the musical instrument hits the ground the string was in rest position and hence $u(x,0) = 0$. However, the string was moving at some velocity at impact ($t = 0$), say $u_t(x,0) = -1$. Find the solution $u(x,t)$ for the shape of the string at time t. BONUS: sketch the solution for enough values of t to understand how the string behaves.

6. A string is being forced with a force $f(x)$. Solve the equation

$$\begin{aligned} u_{tt} &= u_{xx} + f(x) \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 0. \end{aligned}$$

   Write the solution in terms of appropriate Fourier coefficients for $f(x)$.