GM system, K spikes

Recall the construction of a single spike on $[-L, L]$:

$$\begin{cases} 0 = \varepsilon^2 u_{xx} - u + \dfrac{u^2}{v} \\ 0 = v_{xx} - v + \dfrac{u^2}{\varepsilon} \end{cases} \qquad u'(\pm L) = 0 = v'(\pm L)$$

Inner region: $x = \varepsilon y$; $v \sim v_0$; $\boxed{u(x) = v_0\, w(y)}$

where $w_{yy} - w + w^2 = 0$; $w(y) = \frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right)$.

Outer region: $u \sim 0$; $v_{xx} - v \sim 0$, $|x| \gg O(\varepsilon)$

and $\displaystyle\int_{-\delta}^{\delta}\left[v_{xx} - v + \frac{u^2}{\varepsilon}\right] = 0$

choose δ s.t. $\varepsilon \ll \delta \ll 1$

then $\displaystyle v_x(\delta) - v_x(-\delta) \sim -\int_{-\delta}^{\delta}\frac{u^2}{\varepsilon}\,dx \sim -\int_{-\delta/\varepsilon}^{\delta/\varepsilon} v_0^2\, w(y)^2\, dy$

$$\sim v_0^2 \int_{-\infty}^{\infty} w^2$$

So in the outer region, $\begin{cases} v'(0^+) - v'(0^-) \sim -v_0^2 \int w^2 \\ v'' - v = 0, \quad x \neq 0 \\ v'(\pm L) = 0 \end{cases}$

Thus

$$v = \begin{cases} A\cosh(x - L), & x > 0 \\ A\cosh(x + L), & x < 0 \end{cases}$$

$\Rightarrow A\left(-\sinh(L) - \sinh(L)\right) = -v_0^2 \int w^2$

$\Rightarrow \quad v(x) = v_0^2 \int w^2 \dfrac{\cosh(L - |x|)}{2\sinh(L)}$

and $v_0 = v(0) \quad \Rightarrow \quad \boxed{v_0 = \dfrac{2}{\int w^2}\tanh(L)}$

Next we study stability of K spikes on domain of size 2KL.

setting $u(x,t) = u(x) + e^{\lambda t}\varphi(x)$
$v(x,t) = v(x) + e^{\lambda t}\psi(x)$

we get:

$$(L) \quad \begin{cases} \lambda\varphi = \varepsilon^2\varphi_{xx} - \varphi + 2\frac{u}{v}\varphi - \frac{u^2}{v^2}\psi \\ 0 = \psi_{xx} - \psi + 2u\varphi \end{cases}$$

As we will see, (L) admits eigenvalues of $O(1)$ which we will call "Large" eigenvalues, as well as eigenvalues of $O(\varepsilon^2)$, which we will call "small" eigenvalues.

- There are K large and K small eigenvalues for a steady state that consists of K spikes.
- When $K=1$, we will see that both large and small eigenvalues are stable for all L. However, for $K \geq 2$, we will derive an <u>instability threshold</u> L_c, such that K spikes are <u>unstable</u> if $L < L_c$ but are stable if $L > L_c$.
[on the domain of size $2KL$].

Large eigenvalues:

In the inner region, let $y = \varepsilon x$; to leading order we get:

$$\begin{cases} \lambda \Phi \approx \Phi_{yy} - \Phi + 2\omega\Phi - \omega^2 \Psi \\ \Psi_{yy} \sim 0 . \Rightarrow \Psi(y) \sim \Psi_0 \end{cases}$$

Outer region: $\Psi_{xx} - \Psi + \frac{2u\varphi}{\varepsilon} = 0$

Choose $\varepsilon \ll \delta \ll 1$ and integrate $\int_{-\delta}^{\delta}$:

$$\Psi_x(\delta) - \Psi_x(-\delta) \sim -\int_{-\delta}^{\delta} 2\frac{u\varphi}{\varepsilon} \sim -2\int v_0 \, \omega(y) \, \Phi(y) \, dy$$

and matching $\Rightarrow \Psi_0 = \Psi(0)$.

Now $\Psi(0) = v_0 \int \omega \, \Phi(y) \, G(0) = \frac{4}{\int \omega^2} \tanh L \left(\int \omega \varphi\right) G(0)$

where $G(\cancel{x})$ satisfies:

$$\begin{cases} G'' - G = 0 \\ G'(0^+) - G'(0^-) = -1 \end{cases} \qquad (G)$$

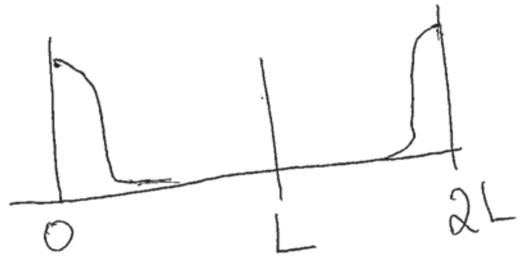
Single spike: Then we solve (G) subject to

$$G'(\pm L) = 0 \quad \Rightarrow \quad G(0) = \frac{1}{2}\coth(L)$$

$$\Rightarrow \quad \lambda \varphi \sim \varphi_{yy} - \varphi + 2\omega\varphi - 2\left(\int \omega \varphi\right) \omega^2 = 0$$

$$\Rightarrow \quad \operatorname{Re} \lambda < 0 \quad [\text{Wei, 1999}]$$

Next consider a double boundary spike configuration:



There are two large eigenvalues in ~~the~~ this case. One of them is even around L; $\Rightarrow$ satisfies $\varphi'(L)=0$

Another is odd at L; satisfies $\varphi(L)=0$

The former is the same as the even eigenvalue of the single spike and is stable.

For the latter, we need to solve (6) subject to $G(\pm L)=0$:

$$G = \begin{cases} A\sinh(x+L), & x<0 \\ A\sinh(L-x), & x>0 \end{cases};$$

$$A(-\cosh(L)-\cosh(L)) = -1$$

$$\Rightarrow G(0) = \frac{1}{2}\tanh(L)$$

$$\Rightarrow \lambda\varphi \sim \varphi_{yy} - \varphi + 2w\varphi - \underbrace{2\tanh^2 L}_{\gamma} \frac{\int w\varphi}{\int w^2} = 0$$

$\Rightarrow$ <u>stable</u> iff $2\tanh^2 L > 1$

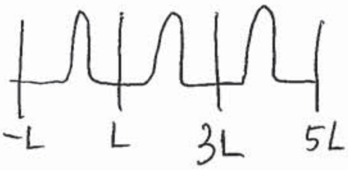
$\Leftrightarrow L > 0.8813$

$$\boxed{\underline{\text{unstable}} \text{ if } L < 0.8813}$$

K spikes, Periodic B.C.:

Consider K-spike configuration on $[-L,(2K-1)L]$

Eg. $\underline{K=3}$:



The linearized pbm. can be written as:

$$P' = M(x)P + \lambda N(x) P \qquad (*)$$

where $P = \begin{bmatrix} \varphi \\ \varphi' \\ \psi \\ \psi' \end{bmatrix}$, and M, N are 4×4

matrices such that $\underset{N}{M}(x+2L) = \underset{N}{M}(x)$

Periodic BC $\Rightarrow$ $P(-L) = P((2K-1)L) \qquad (**)$

Now suppose we can solve $(*)$ subject to

B.C. $P(L) = z P(-L)$ where $z \in \mathbb{C}$.

Then for $x \in [L, 3L]$,

$\underline{\text{set}}$ $P(x) = zP(x-2L)$ Then by periodicity

of N, M, $P(x)$ satisfies $(*)$ on $[-L, 3L]$

and $P(3L) = z^2 P(-L)$

Extending in this way up to $(2K-1)L$,

we get $P((2K-1)L) = z^K P(-L)$

Then $(**)$ is <u>satisfied</u>, provided that $\boxed{z = e^{2\pi i k / K}, \quad k = 0 \ldots K-1}$

So we solve (G) subject to

$$G(+L) = z\, G(-L) \quad ; \quad z = e^{i 2\pi \frac{k}{K}}$$
$$G'(+L) = z\, G'(-L)$$

Then

$$G = \begin{cases} A \cosh(x+L) + B \sinh(x+L) \,, & x<0 \\ zA \cosh(x-L) + zB \sinh(x-L) \,, & x>0 \end{cases}$$

subject to: $\quad G(0^-) = G(0^+) \;;\; G(0^+) - G(0^-) = -1$

$$\Rightarrow \begin{bmatrix} (1-z)\cosh L & (1+z)\sinh L \\ (1+z)\sinh L & (1-z)\cosh L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

After some algebra [HW]:

$$G(0) = \frac{\sinh L \cosh L}{\cosh^2 L + \sinh^2 L - \cos\theta} \,, \quad \theta = 2\pi \frac{k}{K}$$

So we get:

$$\lambda \varphi = \varphi_{yy} - \varphi + 2w\varphi - \gamma \frac{\int w \varphi}{\int w^2} w^2$$

Where $\boxed{\gamma = \dfrac{4 \sinh^2 L}{\cosh^2 L + \sinh^2 L - \cos\theta}}$; $\theta = \frac{2k}{K}$, $k = 0 \ldots K-1$

Small eigenvalues

In terms of $y = \frac{x}{\varepsilon}$, write $u(x) = \mathcal{U}(y)$, $v(x) = V(y)$
$\varphi(x) = \Phi(y)$, $\psi(x) = \Psi(y)$

$$(*) \Rightarrow \begin{cases} \lambda \Phi = \Phi_{yy} - \Phi + 2\frac{\mathcal{U}}{V}\Phi - \frac{\mathcal{U}^2}{V^2}\Psi \\ \Psi_{yy} - \varepsilon^2 \Psi + 2\mathcal{U}\varepsilon\Phi = 0 \end{cases}$$

We expand

$$\lambda = \varepsilon^2 \lambda_0 + \cdots$$
$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \cdots$$
$$\Psi = \varepsilon \Psi_0 + \cdots$$

Then $\Phi_{0yy} - \Phi_0 + 2w\Phi_0 = 0$

Thus $\Phi_0 \approx w_y$ to leading order.

$O(\varepsilon)$:

$$\underbrace{\Phi_{1yy} - \Phi_1 + 2w\Phi_1}_{L\Phi_1} = w^2 \Psi_0 - 2w_y\left(\frac{u_1}{V_0} - \frac{u_0^2}{V_0^2}v_1\right)$$

Recall that $Lw = w^2$ so we may write

$$\Phi_1 = \Psi(0)\,w + \Phi_{1,odd}$$ where $\Phi_{1,odd}$ is odd.

and $\Psi_{0yy} = -2\mathcal{U}\varepsilon w_y$.

Next multiply $(*)$ by $\mathcal{U}_y$ and integrate by parts.

Recall that $\mathcal{U}_{yyy} - \mathcal{U}_y + 2\frac{\mathcal{U}}{V}\mathcal{U}_y - \frac{\mathcal{U}^2}{V^2}V_y = 0$

$$\Rightarrow \quad \lambda \int \Phi\,\mathcal{U}_y = \int \Phi\frac{\mathcal{U}^2}{V^2}V_y - \mathcal{U}_y\frac{\mathcal{U}^2}{V^2}\Psi$$

Estimate: $\int \varphi \frac{U^2}{V^2} V_y - U_y \frac{U^2}{V^2} \Psi$

$$\sim \int \omega^2 \omega_y (V_y - V_0 \Psi) \sim -\int \frac{\omega^3}{3} (V_{yy} - V_0 \Psi_y)$$

$$\sim -\int \frac{\omega^3}{3} (-\varepsilon U^2 - \Psi_y)$$

Now let $F(y) = \int_0^y \frac{\omega^3}{3}$ and write:

$$\int \frac{\omega^3}{3} \Psi_y = \int_{-\infty}^{\infty} F'(y) \Psi_y = \Psi_y F(y) \Big|_{-\infty}^{\infty} - \int F \underbrace{\Psi_{yy}}_{-2U\varepsilon\varphi}$$

$$= \frac{\Psi_y(\infty) + \Psi_y(-\infty)}{2} \int_{-\infty}^{\infty} \frac{\omega^3}{3} - \int F \, \underbrace{2\varepsilon U \omega_y}_{-\varepsilon V_0 \omega^{2\prime}}$$

$$= \langle \Psi_y \rangle \int_{-\infty}^{\infty} \frac{\omega^3}{3} - \int \varepsilon V_0 \omega^2 F' = \int \frac{\omega^3}{3} (\langle \Psi_y \rangle - \varepsilon V_0 \omega^2)$$

$$\Rightarrow -\int \frac{\omega^3}{3} (V_{yy} - V_0 \Psi_y)$$

$$= -\int \frac{\omega^3}{3} \left[\varepsilon^2 V_0 - \cancel{\varepsilon U^2} + \cancel{\varepsilon V_0^2 \omega^2} - V_0 \langle \Psi_y \rangle \right]$$

On LHS we have $\lambda \int \varphi U_y \sim \lambda V_0 \int \omega_y^2$

$$\Rightarrow \boxed{\lambda \int \omega_y^2 \sim \int \frac{\omega^3}{3} \left(\langle \Psi_y \rangle - \varepsilon^2 \right)}$$

where $\langle \Psi_y \rangle = \dfrac{\Psi_y(+\infty) + \Psi_y(-\infty)}{2}$.

Outer region: $\Psi_{xx} - \Psi = -\frac{2u}{\varepsilon}\varphi \qquad (*)$

where $\varphi \sim \omega_y + \varepsilon \Psi_0(0)\omega + \varepsilon\,\varphi_{1odd}, + \cdots$

Note that ω_y is like a dipole (odd) and ω is like a delta fcn (even)

First, take $\varepsilon \ll \delta \ll 1$ and integrate $(*)$ on $[-\delta, \delta]$:

$$\Psi_x(\delta) - \Psi_x(-\delta) \sim -\int 2\varepsilon V_0 \omega^2 \Psi_0(0)\,dy \sim -\Psi_0(0) 2\varepsilon V_0 \int \omega^2$$

Next, multiply $(*)$ by x & then integrate:

- $\int_{-\delta}^{\delta} \Psi_{xx}\, x = \Psi_x\, x \Big|_{-\delta}^{\delta} - \int_{-\delta}^{\delta} \Psi_x = -\Psi(\delta) + \Psi(\delta)$
- $\int -2\frac{u}{\varepsilon}\varphi \underbrace{x\,dx}_{\varepsilon^2 y\,dy} \sim \varepsilon \int -2V_0\,\omega\,\omega_y y\,dy = \varepsilon V_0 \int \omega^2\,dy$

So $(*)$ can we written as:

$$\begin{cases} \Psi_{xx} - \Psi = 0, & x \neq 0 \\ \Psi_x(0^+) - \Psi_x(0^-) = -2\varepsilon V_0 \Psi_0(0) \int \omega^2 \\ \Psi(0^+) - \Psi(0^-) = -\varepsilon V_0 \int \omega^2 \end{cases}$$

Now note that $\Psi_{0_{yy}}(y) = 2\omega\,\omega_y \equiv \text{odd} \Rightarrow \Psi_0 = \Psi_0(0) + \text{odd fcn}$

$$\Rightarrow \Psi_0(0) = \frac{\Psi_0(\infty) + \Psi_0(-\infty)}{2}$$

Also $\Psi_0 = \frac{1}{\varepsilon}\Psi$ and by matching, $\Psi_0(\pm\infty) = \varepsilon\,\Psi(0^\pm)$

Finally, $\dfrac{\partial \Psi}{\partial y} = \varepsilon \partial_x \Psi_x \Rightarrow \langle \Psi_y \rangle = \varepsilon \langle \Psi_x \rangle$

[where $\langle \Psi_y \rangle = \dfrac{\Psi(\infty) + \Psi(-\infty)}{2}$, $\langle \Psi_x \rangle = \dfrac{\Psi(0^+) + \Psi(0^-)}{2}$.

If we let $\eta(x) = \varepsilon \Psi(x)$ then the <u>reduced</u> pblm becomes:

$$\left\{ \begin{array}{ll} \eta'' - \eta = 0 \quad ; & [\eta_x] = -2 V_0 \int \omega^2 \langle \eta \rangle \\ & [\eta] = - V_0 \int \omega^2 \end{array} \right. \qquad (SE1)$$

<u>and</u>

$$\lambda = \frac{\int \omega^3}{3 \int \omega_y} \varepsilon^2 \left(\langle \eta_x \rangle - 1 \right) \qquad (SE2)$$

where $V_0 = \dfrac{2}{\int \omega^2} \tanh L$; $[\eta] = \eta(0^+) - \eta(0^-)$
$\langle \eta \rangle = \dfrac{\eta(0^+) + \eta(0^-)}{2}$

It remains to specify B.C. for η at $\pm L$.

<u>Single spike</u>: Take $\eta'(\pm L) = 0$, and η <u>odd</u>

$$\Rightarrow \langle \eta \rangle = 0; \qquad \eta = \left\{ \begin{array}{ll} A \cosh(x-L), & x > 0 \\ B \cosh(x+L), & x < 0 \end{array} \right.$$

$$-A \sinh L - B \sinh L = 0 \Rightarrow B = -A$$
$$A \cosh L - B \cosh L = - V_0 \int \omega^2 \Rightarrow A = \frac{-V_0 \int \omega^2}{2 \cosh L}$$

$$\Rightarrow \eta = \frac{-V_0 \int \omega^2}{2 \cosh L} \cdot \left\{ \begin{array}{ll} \cosh(x-L), & x > 0 \\ -\cosh(x-L), & x < 0 \end{array} \right.$$

$$\Rightarrow \langle \eta_x \rangle = \tanh L \, \frac{\int \omega^2 V_0}{2} = \tanh^2 L$$

$$\Rightarrow \lambda \sim \frac{\int \omega^3}{3 \int \omega_y^2} (\tanh^2 L - 1) \varepsilon^2$$

$\boxed{\lambda \sim -2 \operatorname{sech}^2(L) \varepsilon^2}$, <u>stable</u> $\forall L, \varepsilon$!

K spikes, periodic BC:

- This is equivalent to taking (SE 1) and imposing B.C.:

$$\boxed{\begin{aligned} \eta(+L) &= z\,\eta(-L) \\ \eta'(+L) &= z\,\eta'(-L) \\ \text{where} \quad z &= e^{i\pi 2\frac{k}{K}}, \quad k = 0 \ldots K-1 \end{aligned}}$$

Then write

$$\eta = \begin{cases} A\cosh(x+L) + B\sinh(x+L), & x<0 \\ zA\cosh(x-L) + zB\sinh(x-L), & x>0 \end{cases}$$

(55)

$$c^2 - s^2 = 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ E \end{pmatrix} \Rightarrow \quad y = -\frac{a}{b}x, \quad cx + dy = E$$

$$\Rightarrow x \qquad \cancel{x} = \frac{bE}{bc-ad} = \frac{-bE}{\det}$$

$$y = \frac{-aE}{bc-ad} = \frac{aE}{\det}$$

Let $c = \cosh L$, $s = \sinh L$;

$$[\eta] = A c(z-1) + B s(-z-1)$$

$$\langle \eta \rangle = A c\left(\frac{z+1}{2}\right) + B s\left(\frac{-z+1}{2}\right)$$

$$[\eta_x] = A s(-z-1) + B c(z-1)$$

$$\langle \eta_x \rangle = A s\left(\frac{-z+1}{2}\right) + B c\left(\frac{z+1}{2}\right)$$

<u>and</u> $[\eta_x] = -4 \tanh L \langle \eta \rangle$

$$[\eta] = -2 \tanh L$$

$$\Rightarrow \underbrace{\begin{bmatrix} s(-z-1) + 2\tanh L\, c(z+1) & c(z-1) + 2\tanh L\, s(-z+1) \\ c(z-1) & s(-z-1) \end{bmatrix}}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} 0 \\ -2\tanh L \end{bmatrix}$$

$$\det = s^2(1+z)^2 - 2\tanh L\, sc(z+1)^2 - c^2(z-1)^2 + 2\tanh L\, sc(1-z)^2$$

$$= -z^2 - 1 + 2z(s^2+c^2) - 8z \tanh L\, sc$$

$$\Rightarrow A = \frac{[c(1-z) + 2\tanh L\, s(-1+z)](-2\tanh L)}{\det}$$

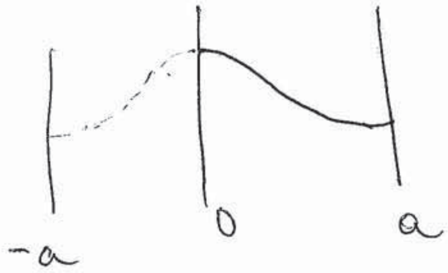
$$B = \frac{[s(-z-1) + 2\tanh L(z+1)c](-2\tanh L)}{\det}$$

After some algebra (HW) we get...

$$\boxed{\langle \eta_x \rangle - 1 = \frac{(\sinh^2 L - 1)(\cos\Theta - 1)}{(\cos\Theta + 2\sinh^2 L - 1)\cosh^2 L}}$$

Neumann BC:

If $\varphi(x)$ is neumann on $[0,a]$, then extend it to $[-a,a]$ by even reflection; it then becomes periodic:



So the eigenvalues of K spikes with Neumann B.C. form a subset of the eigenvalues of 2K spikes with Periodic B.C.

On the other hand, if φ is an eigenfcn on $[-a,a]$ with periodic BC, let

$\hat{\varphi}(x) = \varphi(x) + \varphi(-x)$. Then $\hat{\varphi}'(0) = 0$

and $\hat{\varphi}'(a) = \varphi'(a) - \varphi'(-a) = \varphi'(a) - \varphi'(a) = 0$

$\Rightarrow$ $\hat{\varphi}$ is an eigenfunction on $[0,a]$ with neumann BC, provided that $\hat{\varphi} \not\equiv 0$ on $[0,a]$.

Finally, $\hat{\varphi}$ satisfies a 2-nd order (linear) ODE, and $\varphi'(0)=0$.

Thus $\hat{\varphi} \not\equiv 0 \iff \hat{\varphi}(a) \neq 0 \iff \varphi(a) \neq 0$.

So the eigenvalues of K-spike pattern with neumann BC are:

(SN) Small:
$$\lambda = \frac{2\varepsilon(\sinh^2 L - 1)(\cos\theta - 1)}{(\cos\theta - 1 + 2\sinh^2 L)\cosh^2 L}, \quad \theta = \frac{\pi k}{K}; \quad k = 1 \ldots K$$

(LN) Large: Let $\gamma = \dfrac{4\sinh^2 L}{2\sinh^2 L - (\cos\theta - 1)}$; where $\theta = \dfrac{\pi k}{K}$, $k = 0 \ldots K-1$

Then $\mathrm{Re}(\lambda) < 0 \iff \gamma > 1$.

Note that K-1 small eigenvalues all cross zero simultaneously at $\sinh^2 L = 1 \Rightarrow$

Now let L_θ be sol'n to $\cos\theta - 1 + 2\sinh^2 L = 0$

i.e. $L_\theta = \operatorname{arcsinh}\left(\frac{1-\cos\theta}{2}\right)$

and let $L_c = 0.8813$
$= \operatorname{arcsinh} 1$; i.e. $\sinh^2 L_c = 1$.

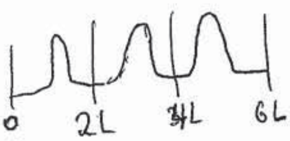
Then:

- $L_\theta < L_c$
- Small eigenvalues are stable if $L > L_c$ or $L < L_\theta$; unstable if $L_\theta \le L < L_c$
- Large eigenvalues are stable if $L > L_\theta$; unstable if $L < L_\theta$.

Summary: Consider GM system

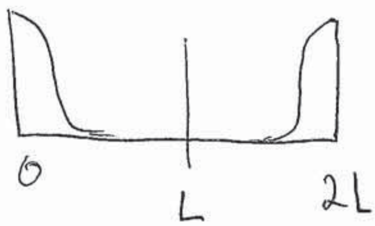
$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + \frac{u^2}{v} \\ 0 = v_{xx} - v + \frac{u^2}{\varepsilon} \end{cases}$$

on the domain of size $2LK$ with Neumann BC. Previously, we have constructed a steady state of K interior spikes by using even reflection of a single spike:

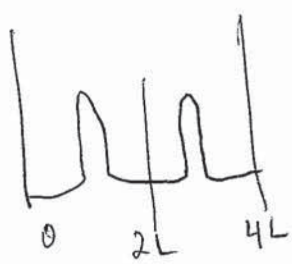


- Suppose that $K=1$ Then a single interior spike is stable $\forall L$.
- Suppose that $K \geq 2$. Let $L_c = \operatorname{arcsinh} 1 = 0.8813$. Then K-spike pattern is stable if $L > L_c$ and is unstable if $L < L_c$.

  At the instability threshold $L = L_c$, K-1 small eigenvalues cross zero simultaneously.

Experiment 1: take $u =$ 

- Unstable if $L < 0.8813$
- Instability comes from $O(1)$ eigenvalue
- If we take $\varepsilon = 0.05$, $L = 1 \Rightarrow$ stable
  $L = 0.8 \Rightarrow$ unstable in $O(1)$ time.

Experiment 2: take $u =$ 

take $\varepsilon = 0.05$

- Small eigenvalues unstable if
  $0.658 < L < 0.8813$
- Large eigenvalues unstable if $L < 0.658$
- Take $L = 1 \Rightarrow$ stable
  $L = 0.8 \Rightarrow$ unstable in $O(\varepsilon^2)$ time
  $\Rightarrow$ very slow instability!!
  $L = 0.6 \Rightarrow$ unstable in $O(1)$ time
  $\Rightarrow$ fast instability

Reference: D.Iron, M.J.Ward and J.Wei, *The Stability of Spike Solutions to the One-Dimensional Gierer-Meinhardt Model,* Physica D, Vol. 150, (2001) pp. 25-62.