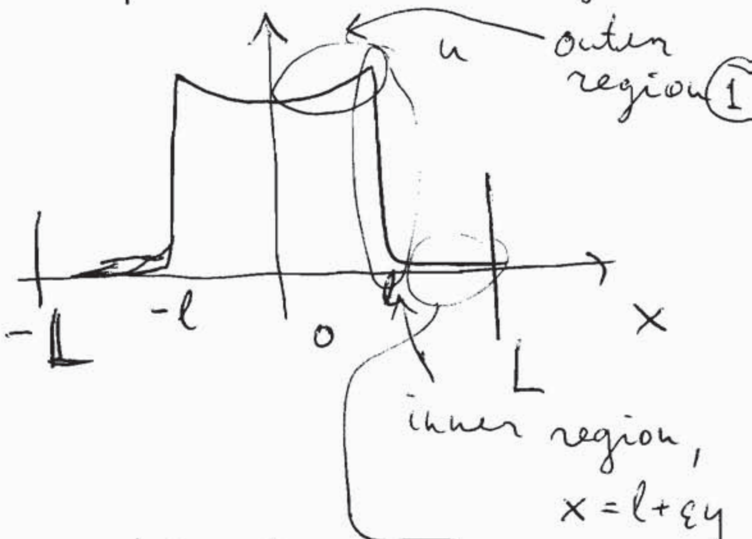
Stability of mesa patterns, $D \geq O(1)$

Again we consider the steady-state solution but with $D \gg O(1)$:

$$\begin{cases} \varepsilon^2 u'' - u + u^2(w-u) = 0 \\ D w'' + 1 - \beta_0 u = 0 \end{cases}$$

If $D = O(1)$ then the mesa pattern is no longer flat on top:



Inner region:

$$x = l + \varepsilon y \;\Rightarrow\; U_{yy} = f(U, W);$$

$u = U(y)$
$w = W(y)$

$$\frac{D}{\varepsilon^2} W_{yy} = \beta_0 U - 1$$

$$\Rightarrow W_{yy} \sim 0 \Rightarrow W \sim W_0 \text{ so as before,}$$

$$\Rightarrow W_0 = \frac{3}{\sqrt{2}} \; ; \; U(y) = U_{h^-}(y) \text{ is}$$

a heteroclinic with $U_{h^\mp}(y) = \frac{1}{\sqrt{2}}\left(1 \mp \tanh\left(\frac{y}{2}\right)\right)$

and with $U(y) \to \sqrt{2}, \quad y \to -\infty$
$U(y) \to 0, \quad y \to +\infty.$

<u>Outer region:</u> u has no sharp gradients so we may ignore the $O(\varepsilon^2 u'')$ term so that

$$-u + u^2(\omega - u) = 0.$$

Note that $\begin{cases} u \to 0 & \text{as } x \to \ell^+ \\ u \to \sqrt{2} & \text{as } x \to \ell^- \end{cases}$

in order to match the inner region.

<u>Outer 2:</u> $u(-1 + u(\omega - u)) = 0 \Rightarrow \boxed{u = 0}$

and ω is quadratic, $D\omega'' = -1$.

<u>Outer 1:</u> $-1 + u\omega - u^2 = 0 \Rightarrow \omega = \frac{1}{u} + u \equiv g(u)$

So u is slave to ω and

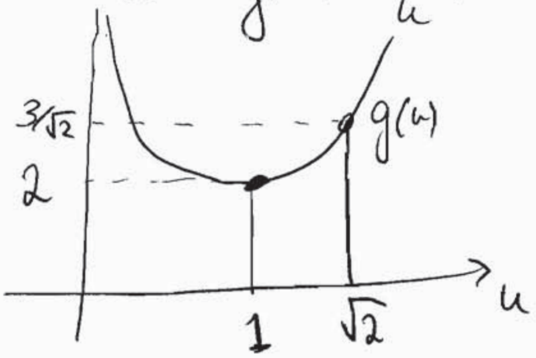
$$\begin{cases} D\omega'' = \beta_0\, g^{-1}(\omega) - 1 \\ \omega'(0) = 0, \quad \omega(\ell) = \frac{3}{\sqrt{2}} \end{cases} \qquad (a)$$

and

$$\int_0^\ell u\, dx = \frac{L}{\beta_0} \qquad (b)$$

The solution to (a) + (b) exists provided that $D > D_c$ for some $D_c = D_c(\beta_0, L)$ [independent of ε]; and it does not exist if $D < D_c$.

In fact note that $w = g(u) = \frac{1}{u} + u$ has the shape



so that $u \in [1, \sqrt{2}]$

$w \in [2, \frac{3}{\sqrt{2}}]$.

Thus

$$w'' = (\beta_0 u - 1)/D$$

$$\geqslant \frac{\beta_0 - 1}{D}$$

$$\Rightarrow w(x) \geqslant w(0) + \frac{\beta_0 - 1}{2D} x^2$$

$$\Rightarrow w(l) \geqslant 2 + \frac{\beta_0 - 1}{2D} l^2$$

On the other hand, $\frac{L}{\beta_0} = \int_0^l u \leqslant l \sqrt{2}$

$$\Rightarrow l^2 \geqslant \frac{L^2}{2\beta_0^2}$$

Thus $\omega(\ell) \geqslant 2 + \frac{\beta_0 - 1}{4\beta_0^2 D} L^2 > \frac{3}{\sqrt{2}}$

provided that $D < \underline{D_c} = \frac{\beta_0 - 1}{4\beta_0^2} L^2 \frac{1}{\frac{3}{\sqrt{2}} - 2}$

So $D_c \geqslant \underline{D_c}$.

Similarly, we have

$$\omega(x) \leq \omega(0) + \frac{(\sqrt{2}\beta_0 - 1)}{D} \frac{x^2}{2} ;$$

$$\omega(0) \geqslant \frac{3}{\sqrt{2}} - \frac{\ell^2}{D} \frac{\sqrt{2}\beta_0 - 1}{2}$$

and $\ell \leq \frac{L}{\beta_0} \Rightarrow -\ell^2 \geqslant -\frac{L^2}{\beta_0^2}$

$$\Rightarrow \omega(0) \geqslant \frac{3}{\sqrt{2}} - \frac{L^2}{D} \frac{\sqrt{2}\beta_0 - 1}{2\beta_0^2} \geqslant 2$$

as long as $D \geqslant L^2 \frac{\sqrt{2}\beta_0 - 1}{2\beta_0^2} \frac{1}{\left(\frac{3}{\sqrt{2}} - 2\right)} \equiv \overline{D_c}$.

Thus we obtain $\underline{D_c} < D_c < \overline{D_c}$.

$$\boxed{\left(\frac{1}{\frac{3}{\sqrt{2}}-2}\right)\frac{\beta_0-1}{4\beta_0^2} \leq \frac{D_c}{L^2} \leq \frac{1}{\left(\frac{3}{\sqrt{2}}-2\right)}\,\frac{\sqrt{2}\,\beta_0-1}{2\beta_0^2}}\,.$$

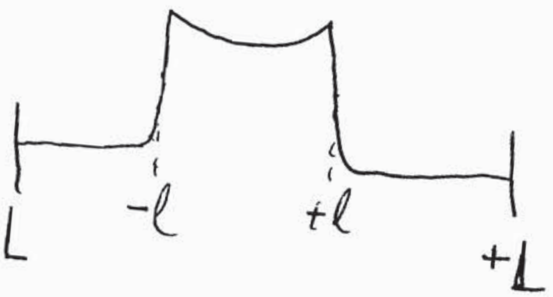
<u>Stability</u>: As before, set

$$u(x,t) = u(x) + e^{\lambda t}\varphi(x)$$
$$w(x,t) = w(x) + e^{\lambda t}\psi(x)$$

and we will also take $\boxed{\tau = 0}$. Then:

$$\begin{cases} \lambda\varphi = \varepsilon^2\varphi_{xx} - f_u(u,w)\varphi - f_w(u,w)\psi \\ \dfrac{\lambda}{\alpha}\,\varphi = D\psi_{xx} - \beta_0\varphi \end{cases}$$

where $D = \dfrac{\varepsilon^2}{\alpha}$. We will consider a single-mesa pattern on $[-L, L]$:



and label interface location $\pm\ell$.

Near $x = \pm\ell$, we rescale,

$$y = \frac{x-(\pm\ell)}{\varepsilon}; \qquad \varphi(x) = \Phi(y)$$
$$\psi(x) = \Psi(y)$$

so that, using $D = \frac{\varepsilon^2}{\alpha}$,

$$\lambda \Phi = \Phi_{yy} - f_u(u,w)\,\Phi - f_w(u,w)\,\Psi$$

$$\frac{\lambda}{\alpha}\Phi = \frac{1}{\alpha}\Psi_{yy} - \beta_0 \Phi \,.$$

Next, expand
$$\Phi = \Phi_0 + \alpha \Phi_1 + \cdots$$
$$\lambda = \alpha\lambda_1 + \cdots$$
$$\Psi = \alpha\Psi_1 + \cdots$$

Then $O(1)$ gives:
$$\Phi_{yy} - f_u \Phi = 0$$

so that $\Phi = C_{\pm}\, u_{h\mp}(y)$

Where $C_{\pm}$ is to be determined

We also expand
$$u = u_0 + \alpha u_1 + \cdots$$
$$W = W_0 + \alpha W_1 + \cdots$$

and we found $u_0 = u_{h\mp}$; $W_0 = \frac{3}{\sqrt{2}}$

The following order equation for Φ is:

$$\lambda_1 \Phi_0 = \Phi_{1yy} - f_u(U_0, W_0)\Phi_1 - f_w(U_0, W_0)\Psi_1$$
$$- \Phi_0 \left(f_{uu}(U_0, W_0) U_1 + f_{uw}(U_0, W_0) W_1 \right)$$

Multiply by U_0' & integrate by parts & use $\Phi_0 = c_\pm U_0'$

$$\text{(a)} \quad \Rightarrow \lambda_1 c_\pm \int U_0'^2 = - \int \Psi_1 f_w U_0'$$
$$- \int \Phi_0 U_0' \left(f_{uu} U_1 + f_{uw} W_1 \right).$$

Now we also have:

$$\text{(b)} \qquad U_1'' - f_u U_1 - f_w W_1 = 0$$

$$\text{(c)} \quad \Rightarrow (U_1')'' - U_0' \left(f_{uu} U_1 - f_{wu} W_1 \right)$$
$$- f_u U_1' - f_w W_1' = 0 .$$

Multiply by Φ_0 & integrate:

$$\text{(d)} \quad - \int \Phi_0 U_0' \left(f_{uu} U_1 - f_{uw} W_1 \right) - \int f_w W_1' \Phi_0 = 0$$

Sub (d) into (a):

$$\boxed{\lambda_1 c_\pm \int \mathcal{U}_0'^2 = c_\pm \int f_w W_1' \mathcal{U}_0' - \int f_w \Psi_1 \mathcal{U}_0'}$$

Now $\quad W = W_0 + \alpha W_1(y)$

$$= w(\pm l + \varepsilon y)$$

$$\sim w(\pm l) + \varepsilon y\, w'(\pm l)$$

$$\Rightarrow W_1(y) = \frac{\varepsilon}{\alpha} y\, w'(\pm l)$$

$$W_1'(y) \sim \frac{\varepsilon}{\alpha} w'(\pm l) .$$

Similarly, $\psi \sim \alpha \Psi_1(y) = \psi(\pm l + \varepsilon y) \sim \psi(\pm l) + \ldots$

$$\Rightarrow \Psi_1(y) \sim \frac{1}{\alpha} \psi(\pm l)$$

So $\quad \displaystyle\int f_w W_1' \mathcal{U}_0' \sim \frac{\varepsilon}{\alpha} w'(\pm l) \int -(\mathcal{U}_0)^2 \mathcal{U}_0'$

$$\sim -\frac{\varepsilon}{\alpha} w'(\pm l) \left. \frac{\mathcal{U}_0^3}{3} \right|_{-\infty}^{\infty}$$

$$\sim \pm \frac{\varepsilon}{\alpha} w'(\pm l) \frac{2^{3/2}}{3}$$

Similarly, $\quad \displaystyle\int f_w \Psi_1 \mathcal{U}_0' \sim \frac{\pm 1}{\alpha} \psi(\pm l)\, w'(\pm l) \frac{2^{3/2}}{3} .$

Finally, $\quad \displaystyle\int_{-\infty}^{\infty} \mathcal{U}_0'^2 = \frac{1}{3}$ $\quad$ [please check this!]

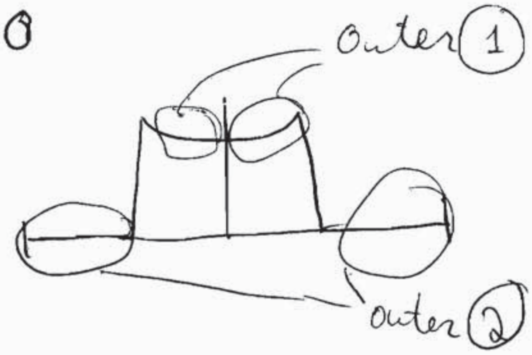
We then obtain,

$$(***) \quad \lambda_1 C_\pm \frac{1}{3} = \pm\frac{2^{3/2}}{3} \qquad \frac{1}{\alpha}\left(\omega'(\pm\ell)\, C_\pm\, \varepsilon - \Psi(\pm\ell)\right)$$

<u>Outer region:</u> Ignore $\varepsilon^2 \varphi_{xx} \ll 1 \Rightarrow$

$$f_u \varphi + f_w \Psi = 0 \Rightarrow \varphi = -\frac{f_w}{f_u}\Psi = \frac{u'}{\omega'}\Psi$$

$$(*) \qquad D\,\Psi_{xx} - (\beta_0 + \lambda_1)\varphi = 0$$



Note that $f(u,w) = 0$

$$f_u u' + f_w \omega' = 0$$

$$\Rightarrow \varphi = \frac{u'}{\omega'}\Psi$$

$$(**) \quad \boxed{D\,\Psi_{xx} - (\beta_0 + \lambda_1)\frac{u'}{\omega'}\varphi = 0}$$

In outer 2, $\frac{u'}{\omega'} \sim 0 \Rightarrow D\,\Psi_{xx} = 0$.

Since $\Psi'(L) = 0$, we obtain $\Psi(x) \equiv \Psi(\ell^+)$, $x > \ell$

$$\Psi(x) \equiv \Psi(-\ell^-), \quad x < -\ell.$$

To determine Ψ in regions 1, integrate (*) over the interfaces:

$$D\,\Psi_x \Big|_{\pm l^-}^{\pm l^+} \sim (\beta_0 + \lambda_1) \int_{\pm l_-}^{\pm l^+} \varphi$$

Recall that near interfaces, we have

$$\varphi(x) \sim C_\pm\, U'_{h^\mp}(y)$$

$$\Rightarrow \int_{\pm l^-}^{\pm l^+} \varphi(x) \sim C_\pm \int U'_{h^\pm}(y)\, \underbrace{dx}_{\varepsilon y}$$

$$\sim \mp\, \varepsilon\, C_\pm \sqrt{2}\ .$$

So near $x = \pm l$ we obtain [using $\Psi_x(\pm l^\pm) = 0$]:

$$D\,\Psi_x(\pm l^\mp) = (\beta_0 + \lambda_1)\, C_\pm\, \varepsilon \sqrt{2}$$

i.e. <u>we need to solve</u>:

(a) $D\,\Psi_x(+l^-) = (\beta_0 + \lambda_1)\, C_+\, \varepsilon\sqrt{2}$

(b) $D\,\Psi_x(-l^+) = (\beta_0 + \lambda_1)\, C_-\, \varepsilon\sqrt{2}$

(c) $D\,\Psi_{xx} - (\beta_0 - \lambda)\, \dfrac{\partial u}{\partial w}\, \Psi = 0\ ,\quad x \in (-l, l)$

Next, let Ψ_r, Ψ_l solve (c) together with

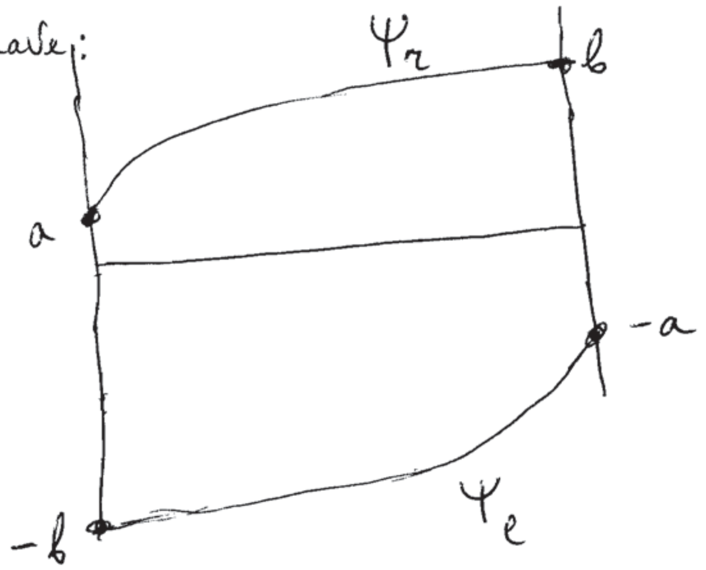
$$\Psi_r'(-l) = [illegible] \quad , \quad \Psi_r'(l) = 0$$

$$\Psi_l'(-l) = 0 \qquad \Psi_l'(l) = [illegible]$$

Note that $\frac{\partial u}{\partial \omega}$ is <u>even function</u>;

thus $\Psi(-x)$ solves (c) $\Longleftrightarrow$ $\Psi(x)$ solves (c)

So we have:



$$\Psi_l = -\Psi_r(-x)$$

Let $a = \Psi_r(-l)$, $b = \Psi_r(l)$; so then

$$-b = \Psi_l(-l) \quad , \quad -a = \Psi_l(l) .$$

In terms of Ψ_ℓ, Ψ_R we obtain:

$$\Psi(x) = \Psi'(\ell)\, \Psi_\ell(x) + \Psi'(-\ell)\Psi_R(x)$$

satisfies (a), (b), (c),

$$\Rightarrow \quad \Psi(\ell) = \Psi'(\ell)(-a) + \Psi'(-\ell)\, b$$

$$\Psi(-\ell) = \Psi'(\ell)(-b) + \Psi'(-\ell)\, a$$

$$\Rightarrow \begin{bmatrix} \Psi(\ell) \\ \Psi(-\ell) \end{bmatrix} = \frac{(\beta_0+\lambda_1)\varepsilon\sqrt{2}}{D} \begin{bmatrix} C_+(-a) + C_- b \\ C_+(-b) + C_- a \end{bmatrix}$$

$$= \frac{(\beta_0+\lambda_1)\,\varepsilon\sqrt{2}}{D} \begin{bmatrix} -a & b \\ -b & a \end{bmatrix} \begin{bmatrix} C_+ \\ C_- \end{bmatrix} .$$

Next, note that $\frac{\varepsilon^2}{\alpha} \approx D = O(1) \Rightarrow \frac{\varepsilon}{\alpha} \gg 1$.

Thus from page 9, eq. (***) we obtain:

$$\Psi(\pm\ell) = \omega'(\pm\ell)\, C_\pm\, \varepsilon$$

By symmetry, $\omega'(\ell) = -\omega'(-\ell)$ and for $x > \ell$ we have: $D\omega'' \sim 1$; $\omega'(L) = 0$

$$\Rightarrow \boxed{\omega'(\ell) = \frac{L-\ell}{D}}$$

So we have:

$$\frac{(\beta_0+\lambda_1)}{D}\,\varepsilon\sqrt{2}\begin{bmatrix}-a & b\\ -b & a\end{bmatrix}\begin{bmatrix}C_+\\ C_-\end{bmatrix} = \begin{bmatrix}\frac{L-\ell}{D}\,C_+\,\varepsilon\\ -\frac{(L-\ell)}{D}\,C_-\,\varepsilon\end{bmatrix}$$

<u>or</u>:

$$(\beta_0+\lambda_1)\sqrt{2}\begin{bmatrix}-a & -b\\ -b & -a\end{bmatrix}\begin{bmatrix}C_+\\ C_-\end{bmatrix} = (L-\ell)\begin{bmatrix}C_+\\ C_-\end{bmatrix}$$

Thus $\dfrac{L-\ell}{(\beta_0+\lambda_1)\sqrt{2}}$ is an eigenvalue of $\begin{bmatrix}-a & -b\\ -b & -a\end{bmatrix}$

These are given by $(-a-\hat{\lambda})^2 - b^2 = 0$

$$\Rightarrow \hat{\lambda} = \pm b - a$$

<u>Summary</u>: λ_1 satisfies

$$\frac{L-\ell}{(\beta_0+\lambda_1)\sqrt{2}} = \pm b - a$$

Where $a = \Psi_R(-\ell)$, $b = \Psi_R(\ell)$, and

$$\begin{cases} D\,\Psi_R'' - (\beta_0+\lambda_1)\,\dfrac{u'}{w'}\,\Psi_R = 0, & x\in(-\ell,\ell)\\ \Psi_R'(-\ell) = 1, \quad \Psi_R(\ell) = 0. \end{cases}$$

Next, let
$$\begin{cases} V_o(x) = \Psi_R(x) + \Psi_\ell(x) \\ \\ V_e(x) = \Psi_\ell(x) - \Psi_R(x) \end{cases}$$

Then we have:
$$V_{\substack{o\\e}}'(\ell) = 1$$
$$V_o(0) = 0 \quad [V_o \text{ is odd}]$$
$$V_e'(0) = 0 \quad [V_e \text{ is even}]$$

and
$$V_o(\ell) = b - a,$$
$$V_e(\ell) = -b - a.$$

<u>Summary</u>: λ_1 satisfies either

$(\lambda_1 \text{ odd})$:
$$\frac{L-\ell}{\sqrt{2}} = V_o(\ell)(\beta_0 + \lambda_1) \quad \text{with}$$
$$\begin{cases} D\, V_o'' - (\beta_0 + \lambda_1)\, \frac{u'}{w'}\, V_o = 0 \\ V_o(0) = 0, \quad V_o'(\ell) = 1 \end{cases}$$

or

$(\lambda_1, \text{ even})$:
$$\frac{L-\ell}{\sqrt{2}} = V_e(\ell)(\beta_0 + \lambda_1)$$
$$D\, V_e'' - (\beta_0 + \lambda_1)\, \frac{u'}{w'}\, V_e = 0$$
$$V_e'(0) = 0, \quad V_e'(\ell) = 1$$

Claim 1: (λ_1, odd) satisfies: $\lambda_1 < 0$

Claim 2:

$$\boxed{\lambda_{1,\text{even}} < \lambda_{1,\text{odd}}}$$

Proof of claim 1: Let $\mu = \beta_0 + \lambda_1$ and let

$$f(\mu) = \mu\, V_0(\ell)$$

Step 1: $f(\mu)$ is increasing for $\mu > 0$

[Remark: we have $\begin{cases} V_0'' - \mu V_0\, h(x) = 0 \\ V_0(0) = 0,\ V_0'(\ell) = 0 \end{cases}$, $h(x) = \frac{u'}{\overline{w}'} > 0$;

Exercise: show that $\mu \to V_0(\ell)$ is decreasing.

So $f(\mu)$ is a product of an increasing and of decreasing function]

Proof: Let $W(x) = \frac{\partial}{\partial\mu}\left(\mu\, V_0(x)\right) = V_0 + \mu \frac{\partial}{\partial\mu} V_0$.

We have

$$V_0'' - \mu h(x) V_0 = 0$$

$$V_{0\mu}'' - h\, W(x) = 0$$

and $W'' = V_0'' + \mu V_{0\mu}''$

$$\Rightarrow W'' - h\mu W = V_0'' = \mu h V_0$$

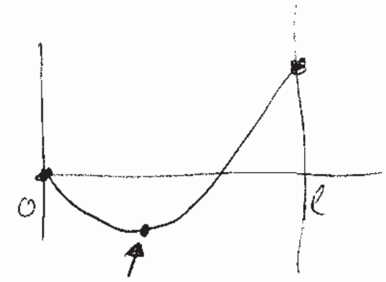
so that:
$$\begin{cases} W(0)=0\ ; & W'(\ell)=1 \\ W''-h\mu W=\mu h V_0 \end{cases}$$

Then let $W_\mu = \frac{\partial}{\partial\mu} W$

$$\Rightarrow \begin{cases} W_\mu(0)=0, & W_\mu'(\ell)=0 \\ W_\mu''-\mu h W_\mu = 2hW \end{cases}$$

By max principle, $V_0 > 0$

[counter-example:



Now suppose that

$W(\ell) \leq 0$. Then since $V_0>0$, $W''-h\mu W > 0$

$\Rightarrow$ we get that $\boxed{W(x)<0 \quad \forall\, x\in(0,\ell)}$ by max principle

[Counter-example: $W''<0,\ W>0 \Rightarrow W''-\mu h W<0$

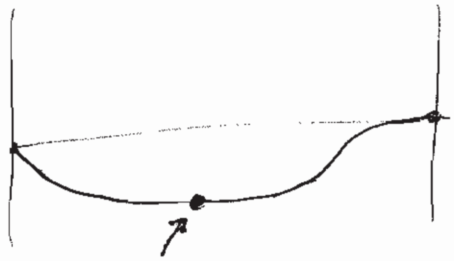
]

But in this case, we have

$$W_\mu''-\mu h W_\mu < 0 \quad , x\in(0,\ell)$$

$$\Rightarrow W_\mu > 0 \quad \text{by max principle}$$

[Counter-example:



$w_\mu'' > 0,\ w < 0$

$\Rightarrow\ w_\mu'' - \mu h w_\mu > 0$]

<u>Conclusion</u>: $f''(\mu) > 0$ whenever $f'(\mu) < 0$.

Thus <u>f has no local max.</u>

To complete the proof of Step 1, it suffices to show that $f'(0) > 0$.

For small μ, we <u>expand</u>: $V_{old} = V_0 + \mu V_1 + \dots$

$$\Rightarrow \begin{cases} V_0'' = 0, & V_0(0) = 0, \quad V_0'(\ell) = 1 \\ \end{cases}$$

$$\Rightarrow V_0 = x \Rightarrow V_0(\ell) = \ell$$

$$\Rightarrow f(\mu) \sim \mu \ell, \quad \mu \ll 1$$

$$\Rightarrow f'(0) = \ell > 0 \quad \blacksquare$$

Step 2: $f(\beta_0) > \dfrac{L-l}{\sqrt{2}}$.

Proof: When $\mu = \beta_0$, we have: $DV_0'' - \beta_0 \dfrac{u'}{w'} V_0 = 0$

Note that $Dw'' - \beta_0 u = 0$

$$\Rightarrow D(w')'' - \beta_0 \frac{u'}{w'} w' = 0$$

So let $V_1 = \dfrac{w'(x)}{w''(l)} = \dfrac{Dw'(x)}{\beta_0\sqrt{2} - 1}$

Then
$$\begin{cases} DV_1 - \beta_0 \dfrac{u'}{w'} V_1 = 0, & V_1(0) = 0 \\ & V_1'(l) = 1 \end{cases}.$$

Comparison principle: $V_0 \geqq V_1$

Then $f(\beta_0) \geqslant \beta_0\left(\dfrac{Dw'(l)}{\beta_0\sqrt{2} - 1}\right) = \dfrac{L-l}{\sqrt{2} - \frac{1}{\beta_0}} > \dfrac{L-l}{\sqrt{2}}$

Proof of comparison principle:

Let $\varphi = V_0 - V_1$; then

$$D\varphi' - \beta_0 \frac{u'}{w'} \varphi = 0, \quad \varphi(0) = 0, \quad \varphi'(0) = 0$$

$\Rightarrow \varphi \geqq 0$

← counter-example

∎

Summary: $f(\mu) = \frac{L-l}{\sqrt{2}}$, $f(\mu) \nearrow$ for $\mu > 0$

and $f(\beta_0) > \frac{L-l}{\sqrt{2}}$

$$\Rightarrow \mu = \lambda_1 + \beta_0 < \beta_0 \Rightarrow \boxed{\lambda_1 < 0}$$

■

Proof of Claim 2: It suffices to show that

$V_e(l) > V_0(l)$ whenever $\mu = \beta_0 + \lambda_1 \geq 0$.

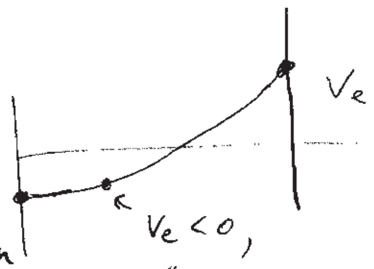
Let $\varphi = V_e - V_0$; $\varphi'' - \mu h \varphi = 0$;

$\varphi'(l) = 0$. If $\varphi(l) < 0$ then

by max principle, $\varphi < 0 \ \forall x$

$\Rightarrow V_e(0) < 0$.

But then:



max principle $\Rightarrow$ contradiction

$V_e < 0$, $V_e'' > 0 \Rightarrow DV_e'' - \mu V_e > 0$

[$V_e > 0$ by max principle]

#

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Reference:
T. Kolokolnikov, M. Ward and J. Wei,
Self-replication of mesa patterns in reaction-diffusion models,
Physica D, Vol.236(2), 2007, Pages 104-122