Multiscale analysis for a PDE with delay

Consider:

(1) $$\begin{cases} u_t = \varepsilon u_{xx} - a\, u(x, t-\tau)\; b\, u^3, & x \in [0, L],\ t \geq 0 \\ \text{Periodic B.C. on } [0, L] \end{cases}$$

If $\varepsilon = 0$: s.s. $u = 0$ has a Hopf bifurcation at $\tau = \tau_0 = \frac{\pi}{2a}$; the linearized sol'n at the Hopf point is $u \sim c\, e^{i\sigma}$, $c \ll 1$, $\boxed{\sigma = a}$.

Q: Take $\varepsilon \ll 1$ and $\boxed{\tau = \tau_0 + \delta \quad \text{where } \delta \ll 1}$. What happens? Can we get inhomogenuities??

[yes if ε is too small, no if ε is sufficiently big]

Multiple scale anzatz:

Take $$\begin{cases} u = \mathcal{U}(x, t, s) & \text{where } s = \delta t \\ b = b_1 \delta \\ \varepsilon = \varepsilon_0 \delta \end{cases} \qquad b_1, \varepsilon_0 = O(1)$$

and expand $\mathcal{U} = \mathcal{U}_0 + \delta\, \mathcal{U}_1 + \dots$

$$u(t-\tau) = U(x, t-\tau_0-\delta, s-\delta\tau_0 - \cdots)$$

$$u_t = U_t + \delta U_s$$

$$u(t-\tau) = U(x,t-\tau_0,s) - \delta U_t(x,t-\tau_0,s) - \delta\tau_0 U_s(x,t-\tau_0$$

$$= U_0(t-\tau) + \delta\big(U_1(t-\tau_0) - U_{0t}(t-\tau_0) - \tau_0 U_{0s}(t-\tau_0)\big)$$

$$\Rightarrow \begin{cases} U_{0t} + aU_0(x,t-\tau_0,s) = 0 \\ U_{1t} + aU_1(x,t-\tau,s) = \varepsilon_0 U_{0xx} - b_1 U_0^3 - U_{0s}(t,s,x) \\ \qquad + aU_{0t}(x,t-\tau_0) + a\tau_0 U_{0s}(t-\tau_0) \end{cases}$$

$O(1)$: $U_0 = A(x, s)e^{iat} + c.c.$ $\quad a\tau_0 = \frac{\pi}{2}$

$$U_0(t-\tau_0) = A(-i)e^{iat}$$

Resonance: $\boxed{\varepsilon_0 A_{xx} - 3b_1 A^2\bar{A} - A_s + a^2 A - i\underbrace{a\tau_0}_{\frac{\pi}{2}} A_s = 0}$

$$(2) \Rightarrow \boxed{\left(1+\frac{\pi}{2}i\right)A_s = \varepsilon_0 A_{xx} + A\left(a^2 - 3b_1|A|^2\right)}$$