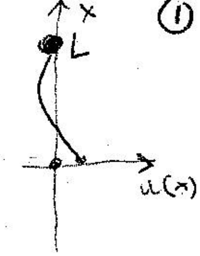


Hanging chain

- Chain attached at $x = L$ to the ceiling
- Tension due to gravity, $T = mgx$
- Let $u(x,t) \equiv$ deviation from vertical position.

Assuming deviation is small, we get:

(1) $$(T u_x)_x = m\, u_{tt} \qquad [\, F = ma \,]$$

Where $m \equiv$ mass density.

Then:
$$\begin{cases} g(x u_x)_x = u_{tt} \\ u(L,t) = 0 \end{cases}$$

Seek sol'n of the form $u(x,t) = \mathcal{U}(x)V(t)$;

then
$$\frac{g(\mathcal{U}_x x)_x}{\mathcal{U}} = \frac{V_{tt}}{V} \Rightarrow \begin{cases} V_{tt} = -\omega^2 V \\ (\mathcal{U}_x x)_x + \frac{\omega^2}{g}\mathcal{U} = \end{cases}$$
(2)

(3) So $V(t) = A\cos(\omega t) + B\sin\omega t$;

Change var:
$$\begin{cases} \mathcal{U}(x) = w(z) \\ z = \frac{\omega^2}{g} x \end{cases}$$
i.e. $\mathcal{U}(x) = w(\frac{\omega^2}{g} x$

(4) $$\Rightarrow (w_z z)_z + w = 0$$

$$(*) \quad \boxed{(w_z z)_z + w = 0}$$

and $\quad u(L) = 0 \Rightarrow \boxed{w\left(\frac{\omega^2}{g} L\right) = 0}$

To solve $(*)$, seek series solution:

$$w(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots = \sum_{n=0}^{\infty} a_n z^n$$

Then $\quad \sum_0^{\infty} a_n n^2 z^{n-1} + \sum_0^{\infty} a_n z^n = 0$

$$\Rightarrow 0 = (a_1 1^2 + a_0) z^0 + (a_2 2^2 + a_1) z^1 + (a_3 3^2 + a_2) z^2 + \dots$$

$$\Rightarrow \begin{cases} a_0 \equiv \text{anything} \\ a_1 = -a_0 \\ a_2 = -\dfrac{a_1}{2^2} = \dfrac{a_0}{2^2} \end{cases}$$

$$a_3 = -\frac{a_2}{3^2} = -\frac{a_0}{(2\cdot 3)^2}$$

$$a_4 = +\frac{a_0}{(2\cdot 3\cdot 4)^2} \dots$$

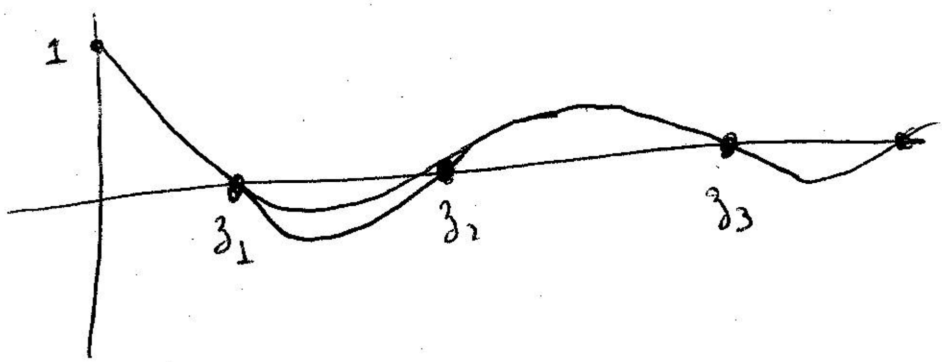
$$\vdots$$

$$\Rightarrow \boxed{w(z) = a_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} z^n \right)} \quad (5)$$

where a_0 is any constant.

Next, note that "a_0" can be absorbed into "A" and "B" in (3) so W.L.O.G., set $a_0 = 1$

Using computer, we can now use (5) to ~~set~~ <u>sketch</u> $w(z)$:



We compute: (Numerically)

$$z_1 = 1.44579$$
$$z_2 = 7.6178$$

etc.

Now $w\left(\frac{\omega^2}{g} L\right) = 0 \Rightarrow \boxed{\omega = \sqrt{\frac{g z_k}{L}}}$

where z_k is a root of $w(z)$.

So $$u(x,t) = \sum \left[A_k \cos(\omega_k t) + B_k \sin(\omega_k t)\right] w\left(\frac{\omega_k^2}{g} x\right)$$

~~Where~~

$$\cancel{= \sum A_k \cos(\omega_k t) + B_k \sin(\omega_k t)}$$

See Maple for example attached.