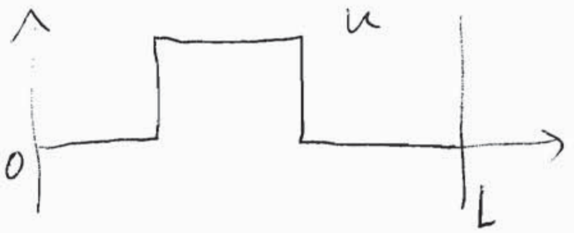
Mesa Patterns in the Brusselator

Consider the <u>Brusselator system</u>:

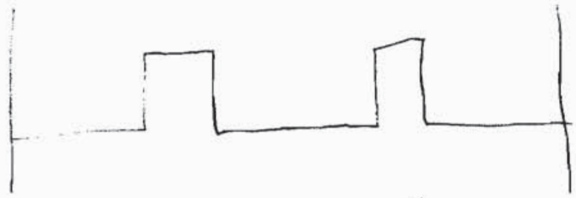
$$\begin{cases} u_t = \varepsilon^2 u_{xx} - u + \alpha + u^2 v \\ \tau v_t = \varepsilon^2 v_{xx} + (1-\beta)u - u^2 v \\ u'(0) = u'(L) \quad ; \quad v'(0) = v'(L) \end{cases}$$

~~We do~~

Numerically, it exhibits <u>mesa-like patterns</u>,



1-mesa pattern



2-mesa pattern.

Let's first construct the steady state equilibrium.

Set $u_t = 0$, $v_t = 0$, and let

$w \equiv v + u$; then $v = w - u$;

$$\varepsilon^2 u_{xx} - u + \alpha + (w-u)u^2 = 0$$

$$\varepsilon^2 w_{xx} - \beta u + \alpha = 0$$

<u>Define</u>: $D = \dfrac{\varepsilon^2}{\alpha}$; $\beta_0 = \dfrac{\beta}{\alpha}$

then $D w_{xx} - \beta_0 + 1 = 0$.

Next, we make the following assumptions:

$$\varepsilon \ll 1 \; ; \quad \alpha \ll 1 ; \quad \frac{\beta}{\alpha} = O(1).$$

We obtain:

$$\boxed{\begin{cases} \varepsilon^2 u_{xx} - u + (w-u)u^2 = 0 \\ D w_{xx} - \beta_0 u + 1 = 0 \end{cases}}$$

where $\varepsilon^2 \ll D$; $\beta_0 = O(1)$; $\varepsilon \ll 1$.

To start with, we make an additional assumption:

$$D \gg 1 .$$

Now expand in $\frac{1}{D}$:

$$u = u_0 + \frac{1}{D} u_1 + \cdots$$

$$w = w_0 + \frac{1}{D} w_1 + \cdots$$

Let $\varepsilon^2 u_{xx} = f(u, w) \equiv u + (u-w)u^2$;

then
$$\begin{cases} \text{(a)} & \varepsilon^2 u_{0\,xx} = f(u_0, w_0) \\ \text{(b)} & w_{0\,xx} = 0 \Rightarrow w_0 \equiv \text{const.} \end{cases}$$

and
$$\begin{cases} \text{(c)} & \varepsilon^2 u_{1\,xx} = f_u(u_0, w_0)\, u_1 + f_w(u_0, w_0)\, w_1 \\ \text{(d)} & w_{1\,xx} = 1 - \beta_0 u_0 \end{cases}$$

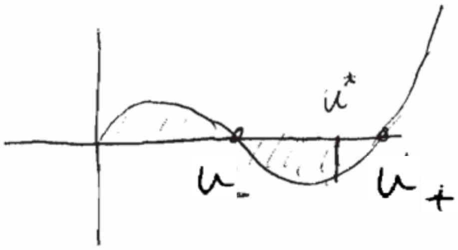
Integrating (d) and using $w_x(0)=0=w_x(L)$;
we obtain: $w_0 \equiv \text{const}$

$$1-\beta_0\int_0^L u_0 = 0$$

so that $\displaystyle\int_0^L u_0 = \frac{1}{\beta_0}$.

Now $f(u,w) = u(1-wu+u^2)$ has
roots $u=0$; $\displaystyle u_\pm = \frac{w\pm\sqrt{w^2-4}}{2}$;

- First assume $w_0>2$ so that f has shape



Let u^* be s.t. $\displaystyle\int_0^{u^*} f(u)\,du = 0$.

Note that $\exists\, u^* \in [u_-, u_+]$ for big enough w_0.

- Suppose $u^* < u_+$. Then to

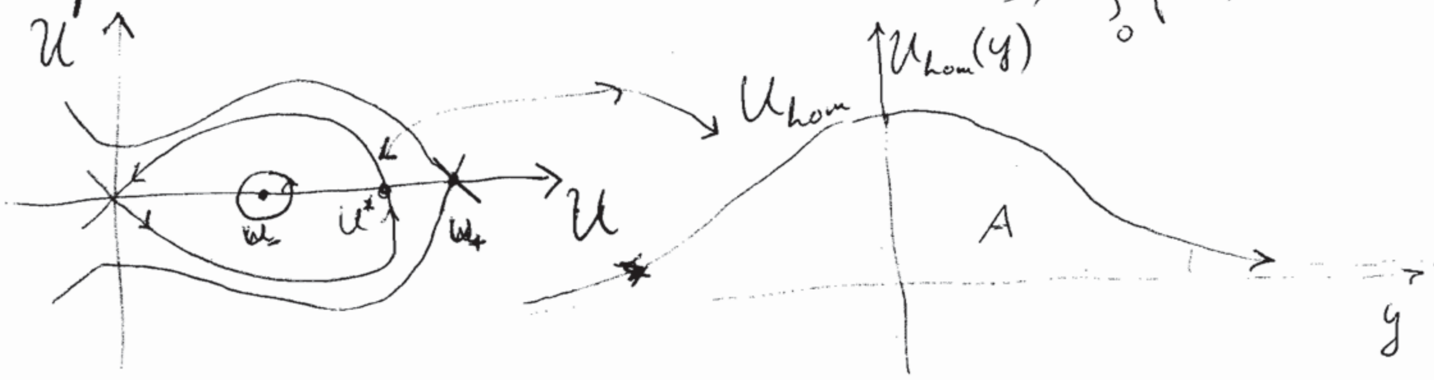
find sol'n to $\begin{cases} \varepsilon^2 u'' = f(u,w_0) \\ u'(0)=0; \quad u'(L)=0 \end{cases}$

we rescale $u(x) = U(y)$; $y = \frac{x}{\varepsilon}$;

$$\Rightarrow \begin{cases} U''(y) = f(U, w_0) \\ U'(0) = 0 \ ; \\ U'(\frac{L}{\varepsilon}) = 0 \end{cases}$$

$\Rightarrow$ $U \sim U_{hom}$ where U_{hom} is the homoclinic connection with $U(0) = u^*$: $\frac{U'^2}{2} = \int_0^U f(s, w_0)\,ds$;

$\Rightarrow \int_0^{u^*} f(s, w_0)\,ds = 0$



Then we have: $U \to 0$ as $y \to \pm\infty$.

Let $A = \int_0^\infty U_h(y)\,dy$;

we have : $\int_0^L u_0(x)\,dx = \varepsilon \int_0^{L/\varepsilon} U(y)\,dy \sim \varepsilon \int_0^\infty U_{hom}(y)\,dy$

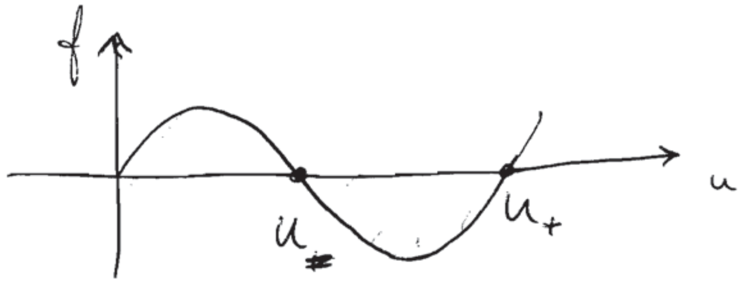
$\sim A\varepsilon = \frac{1}{\beta_0}$

$\Rightarrow$ $\boxed{A = O\left(\frac{1}{\varepsilon}\right)}$

But for this to happen, $\boxed{u^* \sim u_+}$

[so that homoclinic turns into a heteroclinic]

$\Rightarrow$ $f(u, w_0)$ must satisfy the Max-well line condition



Since $f(u, w_0)$ is a cubic in u, this is equivalent to simultaneously solving for w_0:

$$\begin{cases} f(u_-; w_0) = 0 \\ f_{uu}(u_-, w_0) = 0 \end{cases}$$

[i.e. u_- is an inflection point]

$$\Rightarrow \begin{cases} u(1 - wu + u^2) = 0 \\ -2w + 6u = 0 \end{cases}$$

$$\Rightarrow \quad w = 3u; \qquad 1 - 2u^2 = 0;$$

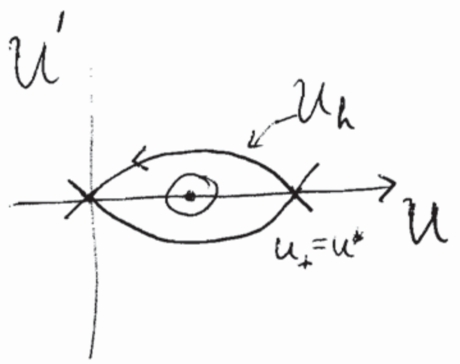
$$u_- = \frac{1}{\sqrt{2}}; \qquad \boxed{w_0 = \frac{3}{\sqrt{2}}; \quad u_+ = \sqrt{2}}$$

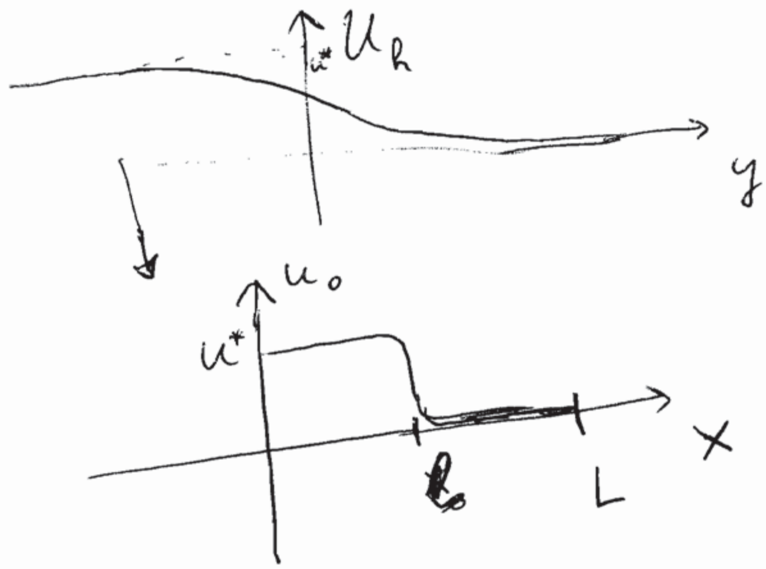
Then $u_0(x) = U_h\left(\frac{x-\ell}{\varepsilon}\right)$; $y = \frac{x-\ell}{\varepsilon}$

Where $U_h(y)$ is a <u>heteroclinic orbit</u>

with $U_h(y) \to 0$ as $y \to \infty$

$U_h(y) \to u_+ = u^*$ as $y \to -\infty$



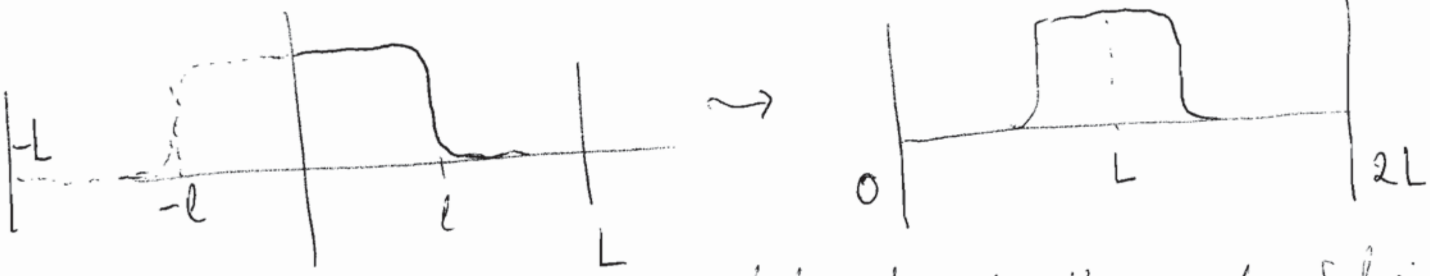


<u>To determine ℓ</u>:

$$\int_0^L u_0 \sim u^* \ell = \sqrt{2}\,\ell = \frac{L}{\beta_0}$$

$$\Rightarrow \boxed{\ell \cong \frac{L}{\sqrt{2}\,\beta_0}}$$

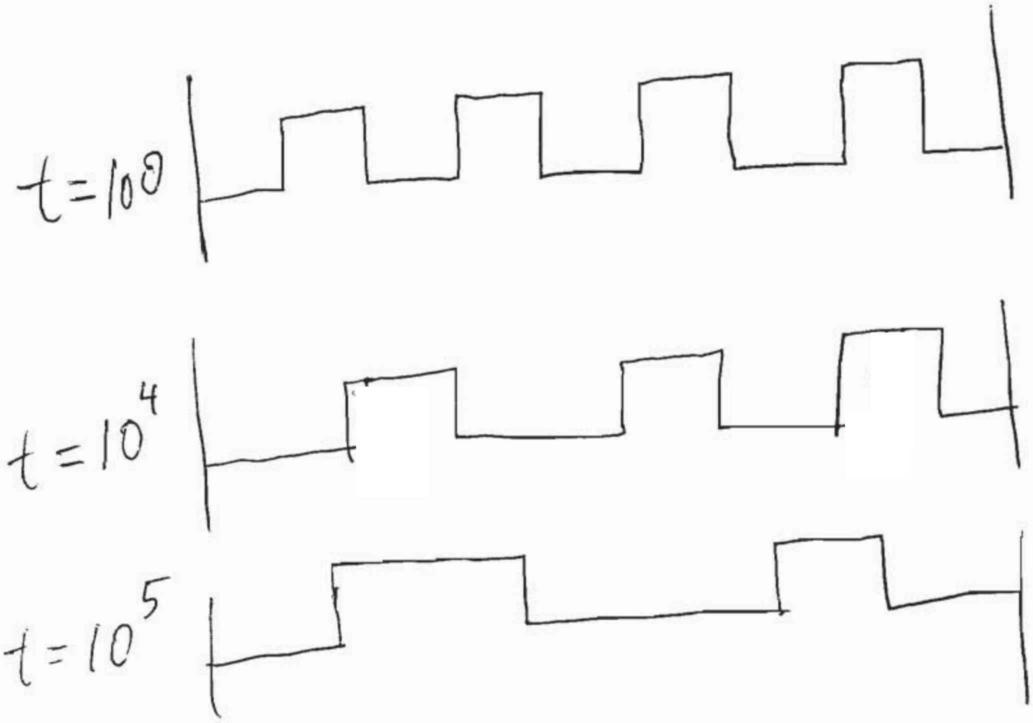
To construct a full mesa, simply use reflection



Note that u; u' is well-defined at the center [glueing pt.] because of Neumann BC. Same for w.

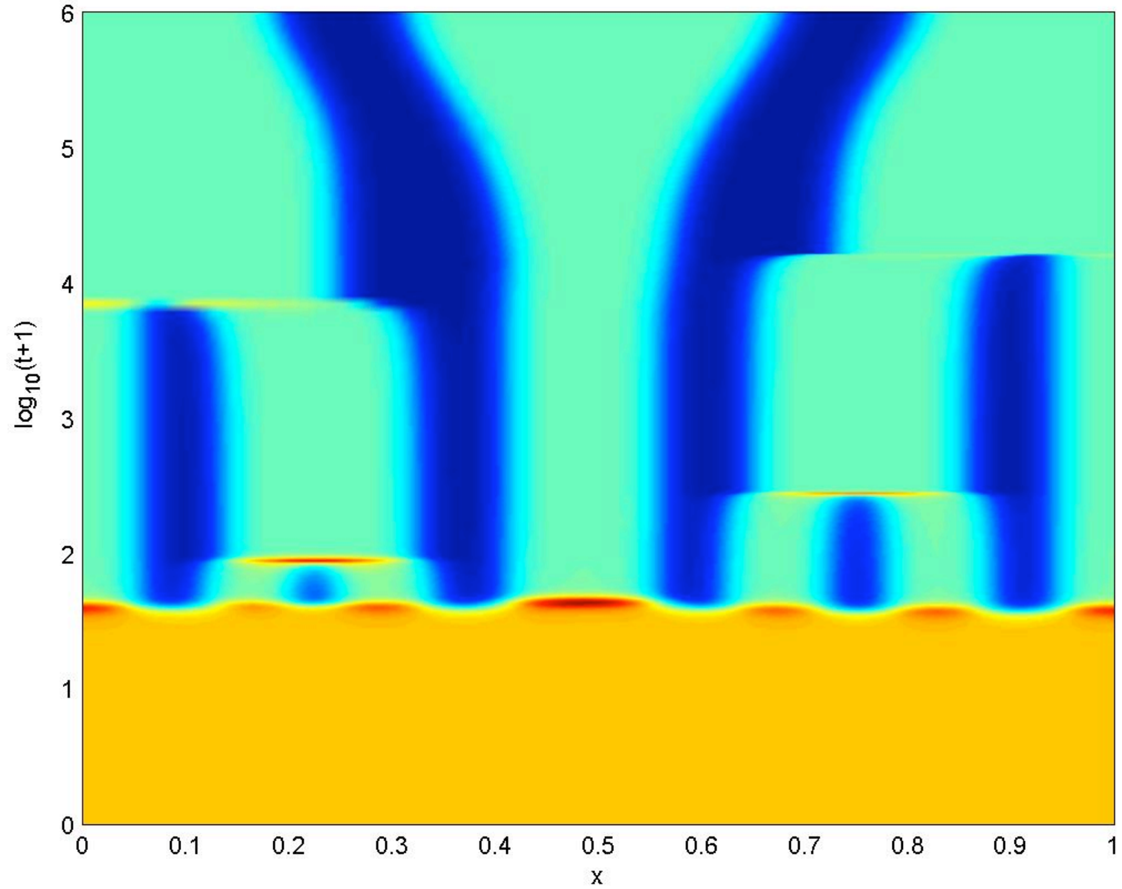
... Similarly for 2, 3, ~~cmes~~ ... K mesa-pattern.

<u>Coarsening phenomenon</u>: Numerically, it is observed that a K-mesa pattern may undergo a <u>coarsening process</u> before reaching a stable equilibrium, as shown below:

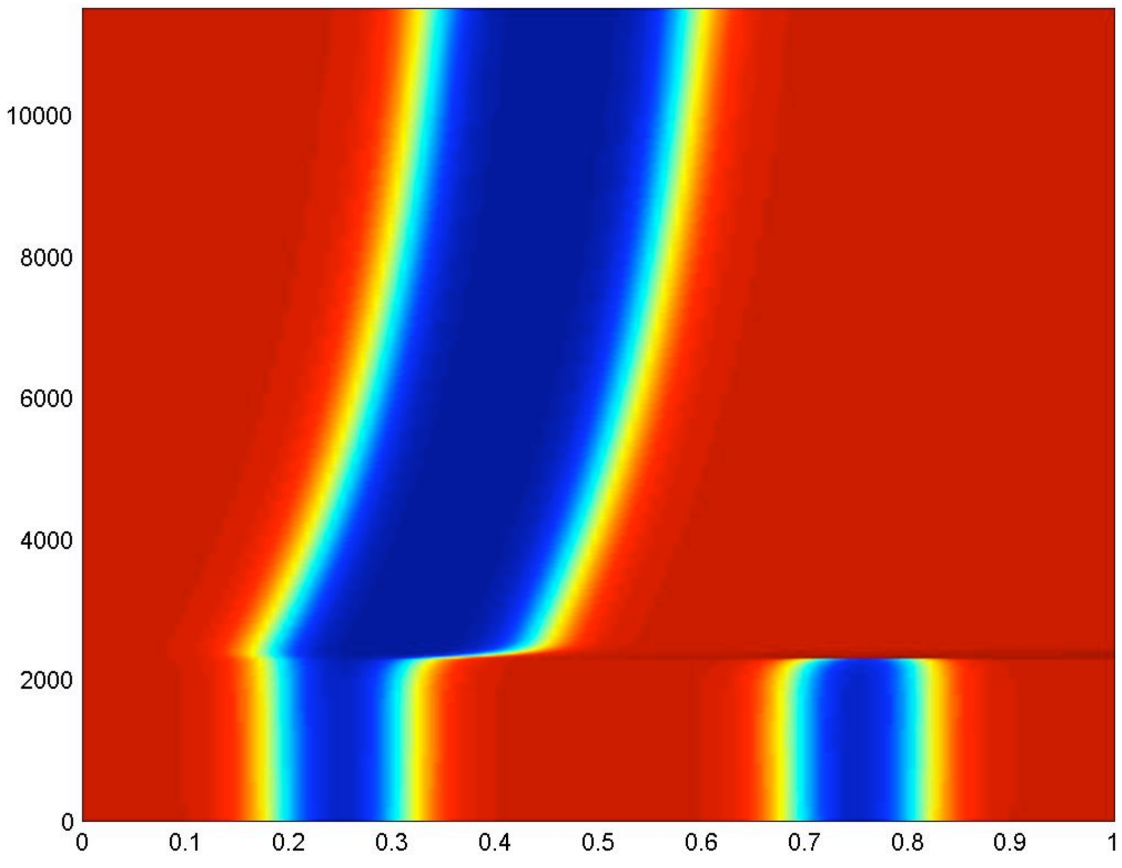


Typically, this occurs on a logarithmically slow timescale, and in the regime when $D \gg 1$.

Below, we will explain this instability by examining the $O(\frac{1}{D})$ terms in the expansion for u, v.



6
5
4
3
2
1
0
log10(t+1)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x



10000
8000
6000
4000
2000
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Asymmetric mesas & mesa splitting:

We will compute w & v to 2 orders.

Again we suppose $D \gg 1$ & expand

$$u = u_0 + \frac{1}{D} u_1 + \cdots$$

$$w = w_0 + \frac{1}{D} w_1 + \cdots$$

$$\Rightarrow \quad \varepsilon^2 u_{0\,xx} = f(u_0, w_0) \qquad w_0(x) \equiv W_0 = \frac{3}{\sqrt{2}}$$

$$u_0(x) = U_h(y) \quad ; \quad y = \frac{x-l}{\varepsilon}$$

$$U_h(y) = \frac{1}{\sqrt{2}}\left(1 - \tanh\left(\frac{y}{2}\right)\right)$$

$$U_h(y) \sim \sqrt{2}\, e^{-y}, \quad y \to \infty$$

$$U_h(y) \sim \sqrt{2}\left(1 - e^{+y}\right), \quad y \to -\infty$$

and:

$$\begin{aligned} &(c) \quad \varepsilon^2 u_{1\,xx} = f_u(u_0, w_0)\, u_1 + f_w(u_0, w_0)\, w_1 \\ &(d) \quad w_{1\,xx} = 1 - \beta_0 u_0 \end{aligned}$$

We use solvability condition from (c):

$$\int \varepsilon^2 u_{1\,xx} u_{0x} - f_u(u_0, w_0) u_{0x} u_1 = \int f_w(u_0, w_0) w_1 u_{0x}$$

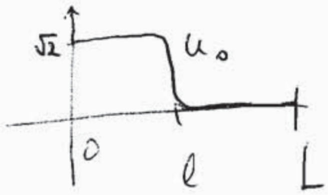
Recall that $f(u_0, w_0) = u_0 + (u_0 - w_0)u_0^2$

$$f_w = -u_0^2$$

So that
$$\int_0^L f_w \, w_1 \, u_{0x} = -\int_0^L \left(\frac{u_0^3}{3}\right)_x w_1$$

$$\sim -w_1(\ell)\,\frac{u_0^3}{3}\Big|_0^L$$

$$\sim +\, w_1(\ell)\,\frac{2^{3/2}}{3}$$



Next, note that

$$\int \varepsilon^2 u_{1\,xx}\, u_{0x} - f_u \, u_{0x} u_1 = \varepsilon^2\left(u_{1x} u_{0x} - u_1 u_{0xx}\right)\Big|_0^L.$$

<u>Near $x \sim L$:</u>

$$u \sim A\left(e^{-\frac{1}{\varepsilon}(x-L)} + e^{+\frac{1}{\varepsilon}(x-L)}\right);$$

$$u_h\left(\frac{x-\ell}{\varepsilon}\right) \sim \sqrt{2}\, e^{-\frac{1}{\varepsilon}(x-\ell)} \sim \sqrt{2}\, e^{\frac{\ell}{\varepsilon}} e^{-\frac{x}{\varepsilon}}$$

$$\sim A e^{\frac{L}{\varepsilon}} e^{-\frac{x}{\varepsilon}} \quad \Rightarrow \quad A = \sqrt{2}\, e^{\frac{-(L-\ell)}{\varepsilon}}$$

and $u \sim u_0 + \frac{1}{D} u_1$

$$\Rightarrow u_0 \sim A e^{-\frac{1}{\varepsilon}(x-L)}\,; \quad \frac{1}{D} u_1 \sim A e^{\frac{1}{\varepsilon}(x-L)}$$

$$\Rightarrow \quad u_{0x}(L) = -\frac{A}{\varepsilon} \qquad u_{1x}(L) = D\frac{A}{\varepsilon}$$

$$u_{0xx}(L) = \frac{A}{\varepsilon^2} \qquad u_1(L) = DA$$

$$\Rightarrow \quad \varepsilon^2 \left(u_{1x} u_{0x} - u_1 u_{0xx} \right) \Big|_{x=L} = -2DA^2$$

$$= -4D e^{-\frac{2(L-l)}{\varepsilon}}$$

<u>Near $x \sim 0$</u>: $u \sim \sqrt{2} - B\left(e^{-\frac{x}{\varepsilon}} + e^{\frac{x}{\varepsilon}}\right)$;

$$u_h\left(\frac{x-l}{\varepsilon}\right) \sim \sqrt{2} - \sqrt{2}\, e^{+\frac{x-l}{\varepsilon}}, \quad x \sim 0$$

$$\Rightarrow u_0 \sim \sqrt{2} - Be^{\frac{x}{\varepsilon}} \quad ; \quad B = \sqrt{2}\, e^{-\frac{l}{\varepsilon}} \quad ;$$

$$u_1 \sim -D\, Be^{-\frac{x}{\varepsilon}}$$

$$\Rightarrow u_1(0) = BD \qquad u_{0xx}(0) = -\frac{B}{\varepsilon^2}$$

$$u_{1x}(0) = \frac{BD}{\varepsilon} \qquad u_{0x}(0) = -\frac{B}{\varepsilon}$$

$$\Rightarrow \quad \varepsilon^2 \left(u_{1x} u_{0x} - u_1 u_{0xx} \right) \Big|_{x=0} = -2B^2 D$$

$$= -4 e^{-\frac{2l}{\varepsilon}} D$$

Thus: $$w_1(\ell) \sim \underbrace{\frac{3}{2^{3/2}}\, 4}_{3\times 2^{1/2}}\, D\left(-e^{-2\frac{(L-\ell)}{\varepsilon}} + e^{-2\frac{\ell}{\varepsilon}}\right)$$

Next, we have $w_{1\,xx} = 1 - \beta u_0$; $w_{1\,x}(0) = 0$

$$\Rightarrow w_1(x) \sim \frac{(1-\beta_0\sqrt{2})}{2}\, x^2 + w_1(0) \, ; \quad x \in (0,\ell)$$

$$\Rightarrow w_1(0) = w_1(\ell) + \frac{\sqrt{2}\,\beta_0 - 1}{2}\,\ell^2 \, ;$$

and recall that $\ell = \frac{L}{\sqrt{2}\,\beta_0}$.

Now suppose that $\beta_0 > \sqrt{2}$;

then $\ell < (L-\ell)$;

$$\Rightarrow -e^{-2\frac{(L-\ell)}{\varepsilon}} + e^{-2\frac{\ell}{\varepsilon}} \sim e^{-2\ell/\varepsilon}$$

and

$$\boxed{w_1(0) \sim 3\times 2^{\frac{1}{2}}\, D\, e^{-\frac{\sqrt{2}}{\beta_0 \varepsilon} L} + \frac{\sqrt{2}\,\beta_0 - 1}{4\beta_0}\, L^2}\,.$$

Note that the two terms balance provided that D is exponentially large.

Also the function $L \to w_1(0)$ attains a minimum at some $L = L^*$.

Let $z = \dfrac{L\sqrt{2}}{\varepsilon \beta_0}$; then $L^2 = \dfrac{\varepsilon^2}{2} \beta_0^2 z^2$

$$w_1(0) \sim A e^{-z} + \varepsilon^2 z^2 \frac{B}{2}$$

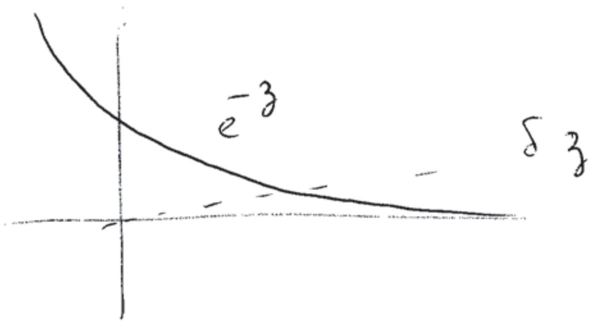
where $A = 3\sqrt{2}\, D$; $B = \dfrac{(\sqrt{2}\beta_0 - 1)\beta_0}{4}$;

and L^* satisfies:

$$e^{-z} = \varepsilon^2 z \frac{B}{A} \; ; \quad z \gg 1$$

Let $\delta = \varepsilon^2 \dfrac{B}{A} = \dfrac{\varepsilon^2 (\sqrt{2}\beta_0 - 1)\beta_0}{12\sqrt{2}\, D} \ll 1$;

$$e^{-z} = \delta z$$



$$-z = \ln(\delta z)$$

$$z = -\ln \delta - \ln z$$

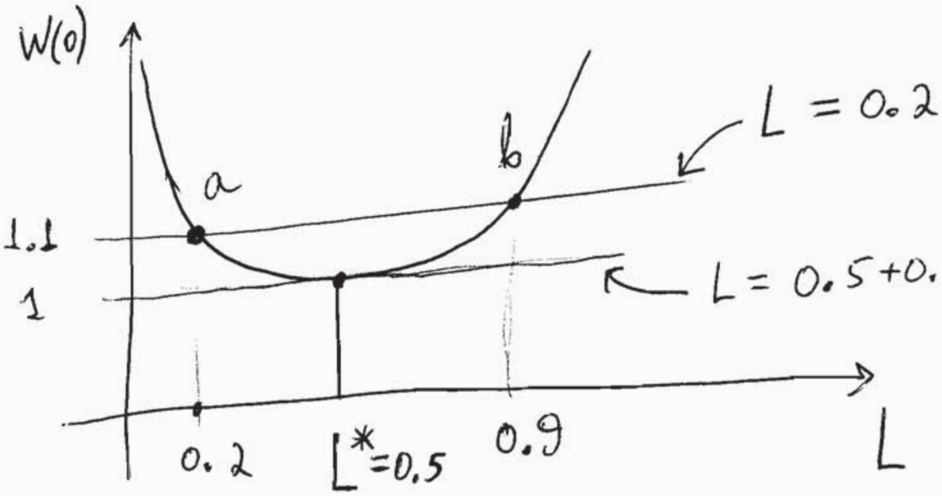
$\Rightarrow$ $z \sim -\ln \delta$ (first-order asymptotics, $\delta \ll 1$)

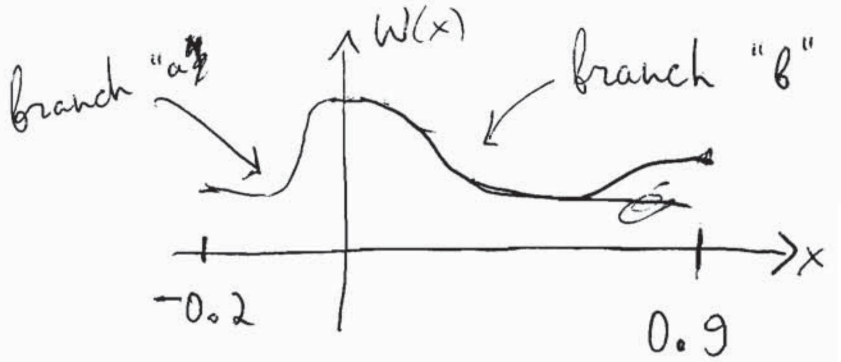
or $z \sim -\ln \delta - \ln \ln \frac{1}{\delta}$ [two-term asymptotic expansion]

$$\Rightarrow \boxed{L^* \sim \frac{\beta_0 \varepsilon}{\sqrt{2}} \ln\left(\frac{12\sqrt{2}\, D}{\varepsilon^2 (\sqrt{2}\beta_0 - 1)\beta_0}\right)}$$

For $L > 2L^*$, an asymmetric mesa pattern can be constructed.

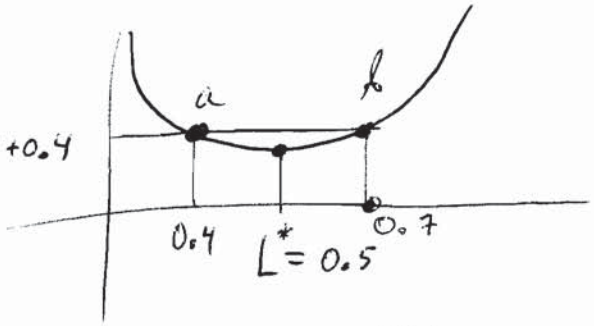
For example, suppose we have:

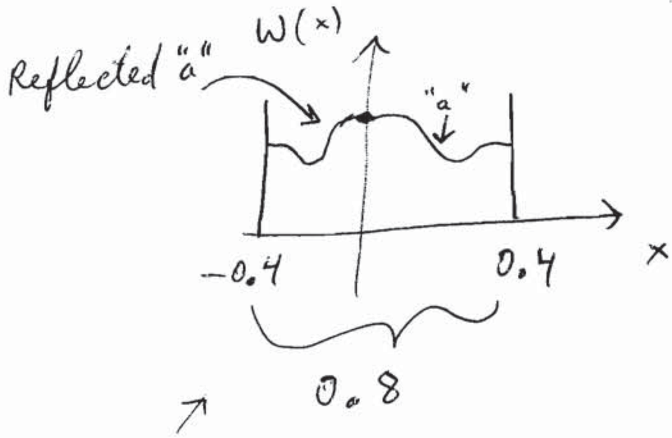




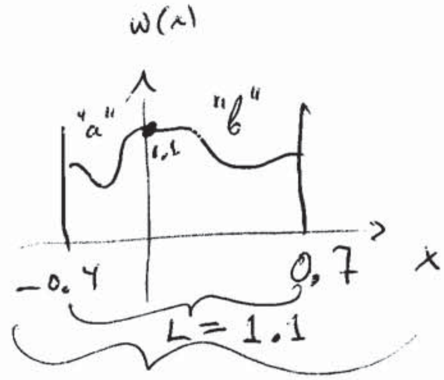
However if $L < 2L^*$, <u>no asymmetric patterns exist:</u>

<u>Example</u>: $L = 0.8 = 0.4 + 0.4$



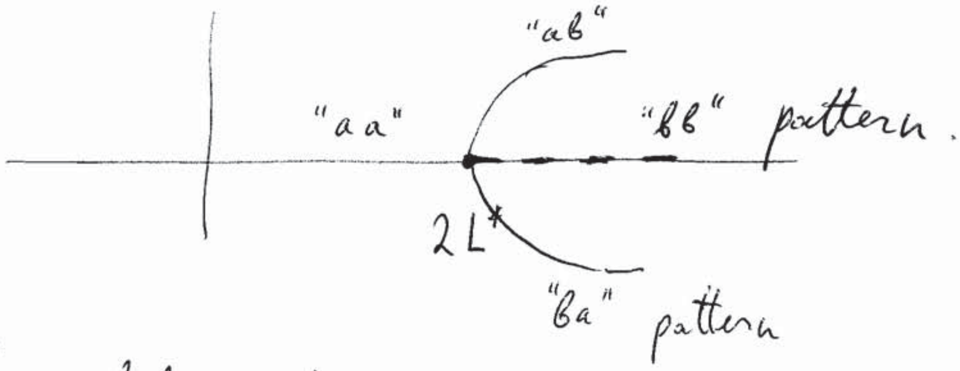


Symmetric pattern ; $L < 2L^*$



Asymmetric pattern; $L > 2L^*$

<u>Conclusion</u>: Asymmetric mesa pattern bifurcates from the symmetric pattern when L is increased past $2L^*$.



<u>Claim</u>:
At the bifurcation point $L = 2L^*$, the eigenvalue of the linearized problem crosses zero.

We have:
$$\begin{cases} u_t = \varepsilon^2 u_{xx} + (w-u)u^2 - u \\ \frac{1}{\alpha} u_t + \frac{\tau}{\alpha}(w-u)_t = D w_{xx} - \beta_0 u + 1 \end{cases}$$

<u>Linearize:</u>
$$\begin{cases} u(x,t) = u(x) + e^{\lambda t}\varphi \\ W(x,t) = w(x) + e^{\lambda t}\psi \end{cases}$$

$$\Rightarrow \begin{cases} \lambda\varphi = \varepsilon^2\varphi_{xx} - f_u\varphi + f_w\psi \\ \frac{\lambda}{\alpha}\left(\varphi + \tau(\psi - \varphi)\right) = D\psi_{xx} - \beta_0\varphi \end{cases}$$

Let
$$u_L = \frac{\partial}{\partial L} u(x;L) \quad ;$$

and let
$$\varphi = u_L' \Big|_{L=L^*}, \quad \psi = w_L' \Big|_{L=L^*}$$

Note that $\varepsilon^2(u')'' - f_u u' - f_w w' = 0$;

and also note that $u_L \Big|_{L=L^*} = \frac{\partial u}{\partial L}\Big|_{L=L^*}$

$$= \frac{\partial u}{\partial s} \underbrace{\frac{\partial s}{\partial L}\Big|_{L=L^*}}_{0} \quad \text{where } s = w(0)$$

$$= 0$$

Thus

$$\varepsilon^2 \varphi'' - f_u \varphi - f_w \psi = 0$$

and similarly, $D \psi_{xx} - \beta_0 \varphi = 0$

Moreover,

$$\varphi'(0) = u_L''(0)\Big|_{L=L^*} = \frac{1}{\varepsilon} \frac{\partial}{\partial L} f(u,w)\Big|_{L=L^*} = 0$$

and similarly, $\varphi'(L) = 0$, $\psi(0) = 0 = \psi(L)$.

So φ, ψ are eigenfunctions and $\lambda = 0$ is an <u>eigenvalue</u>.

Thus λ crosses through 0 at the fold point $L = L^*$.

Similar computation applies if we set $L = 2L^* K$; then the K-symmetric branch becomes unstable.

Thm: Consider a K-mesa symmetric pattern on a domain $[0, L]$ with $\tau = 0$ and suppose $\beta_0 > \sqrt{2}$. Let

$$\boxed{K^* = \frac{L}{\sqrt{2}\,\beta_0\,\varepsilon\,\ln\left(\frac{12\sqrt{2}\,D}{\varepsilon^2(\sqrt{2}\beta_0 - 1)\beta_0}\right)}}\,.$$

This pattern has a zero eigenvalue when $K = K^*$. It is unstable if $K > K^*$ and stable if $K < K^*$.

Proof: So far, we have shown that $\lambda = 0$ when $K = K^*$; see the paper "Mesa-type patterns in the one-dimensional Brusselator and their stability" by T. Kolokolnikov, T. Erneux, and J. Wei, Physica D 214 (2006) 63-77 for the proof of stability when $K < K^*$.